08.128.140 Theoretische Physik 4: Statistische Physik
Theoretical Physics 4: Statistical Physics

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Homework set 8

Due December 18, 2023 by start of lecture.
Please note how long it took you to solve each problem.

8-1, 12 pts. *Differential relations: ordinary vs. partial.* Given that the total energy E in a system is a function of entropy, volume, and number of particles, such that E has the functional dependence $E(S, V, N)$, and given that $dE = TdS - pdV + \mu dN$, express T, p, and μ as partial derivatives of E, indicating which variables are held fixed and which are held constant.

8-2, 20 pts. *Gibbs-Helmholtz equation.* Given that a system has no change in the number of particles and only one displacement variable given by volume, we can write $dE = TdS - pdV$. Defining the enthalpy as $H = E - (-pV)$, the Helmholtz free energy as $F = E - TS$, and the Gibbs free energy as $G = H - TS$, show that

A, 10 pts. $E = -T^2 \left(\frac{\partial}{\partial T}\right)_V \frac{F}{T}$;

B, 10 pts. $H = -T^2 \left(\frac{\partial}{\partial T}\right)_p \frac{G}{T}$.

8-3, 18 pts. *Blundell and Blundell, Problem 16.2. Practice with Maxwell's relations.* Derive the following general relations

A, 6 pts.

$$\left(\frac{\partial T}{\partial V}\right)_E = -\frac{1}{C_V}\left[T\left(\frac{\partial p}{\partial T}\right)_V - p\right] \; ; \tag{1}$$

B, 6 pts.

$$\left(\frac{\partial T}{\partial V}\right)_S = -\frac{1}{C_V} T \left(\frac{\partial p}{\partial T}\right)_V \; ; \tag{2}$$

C, 6 pts.

$$\left(\frac{\partial T}{\partial p}\right)_H = \frac{1}{C_p}\left[T\left(\frac{\partial V}{\partial T}\right)_p - V\right] \; . \tag{3}$$

The variables C_V and C_p are the heat capacity at constant volume and pressure, respectively, and can be written $C_V = \left(\frac{\partial Q}{\partial T}\right)_V$ and $C_p = \left(\frac{\partial Q}{\partial T}\right)_p$, where Q is the heat transfer to the system.

8-4, 30 pts. *Blundell and Blundell, Problem 15.5. Relative entropy.* The relative entropy (also called the Kullback-Leibler divergence) measures the closeness of two probability distributions P and Q and is defined by

$$S(P||Q) = \sum_i P_i \log\left(\frac{P_i}{Q_i}\right) = -S_p - \sum_i P_i \log Q_i \ , \tag{4}$$

where $S_p = -\sum_i P_i \log P_i$ and i labels the events.

A, 15 pts. Show that $S(P||Q) \geq 0$ with equality if and only if $P_i = Q_i$ for all i.

B, 15 pts. If i takes N values with probability P_i, then show that

$$S(P||Q) = -S_P + \log N \ , \tag{5}$$

when $Q_i = 1/N$ for all i. Equivalently, show that $S_P \leq \log N$, with equality if and only if P_i is uniformly distributed between all N outcomes.

8-5, 20 pts. *Blundell and Blundell, Problems 16.6. Hint: review exercise 16.6 in Blundell and Blundell. You can freely use the Maxwell's relations listed in equations (16.48)-(16.51) in Blundell and Blundell.* Show that the entropy per mole of an ideal gas can be expressed as $S = C_p \ln T - R \ln p +$ const., where R is the ideal gas constant.

8-6, Bonus, 10 pts *Information, Monty Hall problem.* After reading example 15.6 in Blundell and Blundell and exercise 15.6 in Blundell and Blundell, explain why the contestant in the game show described in exercise 15.6 should always switch their door choice when given the option by the host. In particular, be specific about what information is different about the choice of the contestant after the host opens the door versus the original random choice.