08.128.140 Theoretische Physik 4: Statistische Physik
Theoretical Physics 4: Statistical Physics

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Homework set 7

Due December 11, 2023 by start of lecture.
Please note how long it took you to solve each problem.

7-1, 20 pts. *H-theorem.* Following Kardar, pp. 72-73, prove the H-theorem. In particular, justify the prefactor of 1/4 in Eq. (3.48) of Kardar from the use of dummy variables and relabeling. *Aside: as a refresher, review how to solve the Gaussian integral using dummy variables.*

7-2, 25 pts. *Ergodicity.* We have seen the ergodic hypothesis come into play a few times during the course, most recently in the construction of probability density functions for Hamltionian time evolution of phase space functions.

A, 15 pts. Write a paragraph identifying the two distinct types of averaging that are related by the ergodic hypothesis.

B, 10 pts. In practice, any realistic experimental apparatus has a finite resolution in time for any measurement (think about pressure as force per unit area needing to be measured for some finite time period, for example). How does this coarse-graining effect justify the use of the ergodic hypothesis?

7-3, 25 pts. *The Boltzmann equation.* The Boltzmann equation addresses the time-evolution of phase space densities via collision terms, and the function $H(t)$ provides a measure of how close a phase space density approaches equilibrium. *Aside: the Boltzmann equation is often written in Hamiltonian formulation as $L[f] = C[f]$, where L is the Liouville operator and C is the collision operator.* Demonstrate that the equilibrium condition dictated by setting $dH/dt = 0$ reproduces the Maxwell-Boltzmann velocity distribution. *Hint: Follow pp. 75-76 of Kardar.*

7-4, 30 pts. *Kardar, Problem 2.6: Information and entropy.* Consider the velocity of a gas particle in one dimension, $-\infty < v < \infty$.

A, 10 pts. Find the unbiased probability density $p_A(v)$ subject only to the constraint that the average *speed* is c, that is, $\langle |v| \rangle = c$. Explicitly, extremize the entropy $S = -\langle \ln \rho \rangle$ imposing constraints of normalization and given average speed using the method of Lagrange multipliers. The starting point is

$$S = -\langle \ln \rho \rangle = -\int_{-\infty}^{\infty} p(v) \ln p(v) dv + \alpha \left(1 - \int_{-\infty}^{\infty} p(v) dv \right) + \beta \left(c - \int_{-\infty}^{\infty} p(v)|v| dv \right) , \tag{1}$$

where α and β are Lagrange multipliers for normalization and average speed, respectively.

B, 10 pts. Find the unbiased probability density $p_B(v)$ subject now only to the constraint that the average kinetic energy $\langle mv^2/2 \rangle = mc^2/2$ for a fixed constant speed c. You will still have to impose normalization as a constraint.

C, 10 pts. Which of the two statements from part A and part B provides more information on the velocity? Quantify the difference in information in terms of $I_B - I_A \equiv (\langle \ln p_A \rangle - \langle \ln p_B \rangle) / \ln 2$.