08.128.140 Theoretische Physik 4: Statistische Physik
Theoretical Physics 4: Statistical Physics

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Homework set 6

Due December 4, 2023 by start of lecture.
Please note how long it took you to solve each problem.

6-1, 20 pts. *Time-evolution of ensemble average and conserved quantities.*

A, 10 pts. Show that, for an ensemble average $\langle \mathcal{O} \rangle$, $\frac{d\langle \mathcal{O} \rangle}{dt} = \langle \{\mathcal{O}, \mathcal{H}\} \rangle$, where $\{\mathcal{O}, \mathcal{H}\}$ is the Poisson bracket of $\mathcal{O}$ and $\mathcal{H}$.

B, 10 pts. Given a conserved quantity L that satisfies $\{L, H\} = 0$, show that $\frac{dL(\vec{p},\vec{q})}{dt} = 0$, where $\vec{p}(t)$ and $\vec{q}(t)$ are the $3N$ conjugate coordinates and coordinates of the system.

6-2, 35+5 pts. *Kardar, problem 3.2. Evolution of entropy.* We consider the entropy $S(t)$ associated to the normalized ensemble density, which is the negative of the expectation value of $\ln \rho(\Gamma, t)$,

$$S(t) = -\langle \ln \rho(\Gamma, t) \rangle \ . \tag{1}$$

A, 10 pts. Show that if $\rho(\Gamma, t)$ satisfies Liouville's equation for a Hamiltonian $\mathcal{H}$, we have $dS/dt = 0$.

B, 15 pts. Using the method of Lagrange multipliers, impose the two constraints of $\int d\Gamma \rho(\Gamma) = 1$ and $\langle \mathcal{H} \rangle = \int d\Gamma \rho(\Gamma)\mathcal{H} = E$ to derive the function $\rho_{\max}(\Gamma)$ that maximizes the functional $S[\rho]$. *Hint: Justify that the modified expression for entropy, $S(t) = \int d\Gamma \rho(\Gamma) \left[-\ln \rho(\Gamma) - \alpha - \beta\mathcal{H}\right] + \alpha + \beta E$ implements the constraints via Lagrange multipliers α and β and extremize $S(t)$ with respect to $\rho(\Gamma)$ accordingly.*

C, 10 pts. Show that the $\rho_{\max}$ derived in part B is stationary.

Bonus, 5 pts. Discuss how part A and part B are not in contradiction, given that part B indicates an increase in entropy as the system approaches the equilibrium density. *Hint: read through problem 3.1 of Kardar, or equivalently, think about equilibrium on small scales versus large scales and the extensive quality of entropy.*

6-3, 25 pts. *Kardar, problem 6.3. The Vlasov equation.* Suppose the number of particles per unit volume and/or the interparticle interaction range in the BBGKY hierarchy is large. This is the opposite limit compared to the dilute limit that leads to the Boltzmann equation and gives an approximate BBGKY equation of

$$\left[\frac{\partial}{\partial t} + \frac{\vec{p}}{m} \cdot \frac{\partial}{\partial \vec{q}} - \frac{\partial U_{\text{eff}}}{\partial \vec{q}} \cdot \frac{\partial}{\partial \vec{p}}\right] f_1\left(\vec{p}, \vec{q}, t\right) = 0 \ , \tag{2}$$

with

$$U_{\text{eff}}(\vec{q},t) = U(\vec{q}) + \int d\mathbf{x}' \mathcal{V}(\vec{q}-\vec{q}\,')f_1(\mathbf{x}',t) \ , \tag{3}$$

and we have reduced the problem of the N-body density to a product of one-particle densities f_1 where $\mathbf{x}_i \equiv (\vec{p}_i, \vec{q}_i)$. Consider N particles confined to a box with volume V, with no additional potential. Show that $f_1(\vec{q},\vec{p}) = g(\vec{p})/V$ is a stationary solution to the Vlasov equation for any $g(\vec{p})$. Why is there no relaxation toward equilibrium for $g(\vec{p})$?

6-4, 20 pts. *Effusion.* Effusion is the process of gas escaping a small hole. Given the gas obeys a Maxwell-Boltzmann velocity distribution and correspondingly the ideal gas law, the molecular flux (number of molecules per unit area per unit time) can be written as

$$\Phi = \frac{p}{\sqrt{2\pi m k_B T}} \ . \tag{4}$$

A, 5 pts. Evaluate the molecular flux of CO_2 gas at standard pressure and temperature (1 atm and 0°C).

B, 15 pts. Given that the probability distribution of molecules effusing out of a hole in some time interval is proportional to $v^3 \exp\left[\frac{1}{2}mv^2/(k_B T)\right]$, where $0 \leq v < \infty$, show that the mean kinetic energy of effusing gas molecules is $\langle \text{K.E.} \rangle = 2k_B T$.