08.128.140 Theoretische Physik 4: Statistische Physik
Theoretical Physics 4: Statistical Physics

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Homework set 5

Due November 27, 2023 by start of lecture.
Please note how long it took you to solve each problem.

5-1, 30 pts. *Moments and cumulants example: Gaussian distribution, relation to central limit theorem.* Given the definition of the Gaussian distribution,

$$p(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[-\frac{(x-\lambda)^2}{2\sigma^2}\right] \tag{1}$$

and given the following moments of the Gaussian distribution,

$$\begin{gathered} \langle x \rangle = \lambda \ , \quad \langle x^2 \rangle = \sigma^2 + \lambda^2 \ , \\ \langle x^3 \rangle = 3\sigma^2\lambda + \lambda^3 \ , \quad \langle x^4 \rangle = 3\sigma^4 + 6\sigma^2\lambda^2 + \lambda^4 \ , \end{gathered} \tag{2}$$

demonstrate that

A, 15 pts. the first cumulant $\langle x \rangle_c = \lambda$, the second cumulant $\langle x^2 \rangle_c = \sigma^2$, and the third and fourth cumulants $\langle x^3 \rangle_c = \langle x^4 \rangle_c = 0$. You can use Eq. (2.11) from Kardar without proof.

B, 15 pts. Write a few sentences justifying the central limit theorem. In particular, for a sum of random variables $X = \sum_{i=1}^{N} x_i$ with each x_i governed by a PDF, construct a related sum $y = (X - N\langle x \rangle_c)/\sqrt{N}$ and justify that y behaves as a Gaussian distribution in the large N limit. *Aside: you can assume that the random variables are bounded such that the variance is finite.*

5-2, 25 pts. *Maxwell-Boltzmann distribution, rms speeds.* The normalized Maxwell-Boltzmann distribution is

$$f(v)dv = \frac{4}{\sqrt{\pi}} \left(\frac{m}{2k_BT}\right)^{3/2} v^2 dv \exp\left[-\frac{1}{2}mv^2/(k_BT)\right] \ , \tag{3}$$

for speeds v between 0 and ∞. Given the root mean squared speed is

$$v_{\rm rms} = \sqrt{\langle v^2 \rangle} = \sqrt{\frac{3k_BT}{m}} \ , \tag{4}$$

A, 5 pts. give the $v_{\rm rms}$ for a hydrogen molecule H_2 at room temperature, $T = 300$ K;

B, 5 pts. give the v_{rms} for a helium atom at the LNe temperature, $T = 77$ K;

C, 5 pts. and give the v_{rms} for an oxygen molecule O_2 in a hot oven, $T = 500$ K.

D, 10 pts. Finally, calculate the mean speed $\langle v \rangle$ and inverse speed $\langle 1/v \rangle$ and show $\langle v \rangle \langle 1/v \rangle = \frac{4}{\pi}$.

5-3, 15 pts. *Deriving the ideal gas law from the Maxwell-Boltzmann distribution.* We can use the Maxwell-Boltzmann distribution to derive the ideal gas law by relating the phase space volume of particles to the pressure of a gas. Since the Maxwell-Boltzmann distribution represents a normalized probability distribution in speeds, we have to augment the probability distribution by probability weights in angular space. Since the total solid angle is 4π, the number of particles in a given v to $v + dv$ and θ to $\theta + d\theta$ is $nf(v)dv\frac{1}{2}\sin\theta d\theta$, where $f(v)$ is the Maxwell-Boltzmann distribution from problem 5-2. *See Figure 6.4 reproduced from Blundell and Blundell.* The flux of particles hitting the wall with unit area and in unit time is $v\cos\theta$. *See the discussion in Blundell and Blundell, Chapter 6.1 and 6.2.*

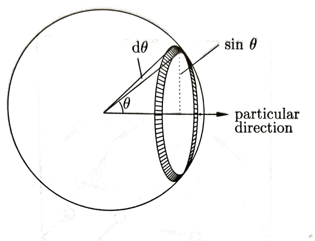

Fig. 6.4 The area of the shaded region on this sphere of unit radius is equal to the circumference of a circle of radius $\sin\theta$ multiplied by the width $d\theta$ and is hence given by $2\pi\sin\theta\, d\theta$.

A, 5 pts. What is the change in momentum of a molecule in the given volume element from hitting the wall? *Hint: it is easiest to apply the law of reflection.*

B, 5 pts. Calculate the pressure as the product of the change in momentum multiplied by the number of molecules per unit volume ($= v\cos\theta\ nf(v)dv\frac{1}{2}\sin\theta d\theta$). Be sure to use and justify the correct integration limits on θ.

C, 5 pts. Rewriting $n = N/V$ where N is the total number of gas molecules in the total volume V, show that your answer in part B reproduces the ideal gas law.

5-4, 10 pts. *Dalton's law, partial pressure.* One consequence of the derivation of pressure for an ideal gas is the fact that for multigas systems in thermal equilibrium, pressures are additive. This is known as Dalton's law, where $n = \sum_i n_i$ is the total number density of each species i, and $p_i = n_i k_B T$ is the partial pressure. Given that air is composed of 75.5% N_2, 23.2% O_2, 1.3% Ar, and 0.05% CO_2 by mass, calculate the partial pressure of oxygen at one atmosphere of pressure and convert your answer to kPa using 1 atm = 101.325 kPa.

5-5, 20+5 pts. *Basic collision theory, hard sphere scattering.* Consider a molecule with cross-sectional area σ with a expected relative speed $\langle v_r \rangle \approx \sqrt{2}\langle v \rangle$. We can define $P(t)$ = the probability of a molecule not colliding up to time t as $P(t) \propto \exp[-n\sigma\langle v_r \rangle t]$, with n being the molecular number density and $P(0) = 1$.

A, 5 pts. What is the proper normalization factor for $P(t)$?

B, 15 pts. Calculate the *mean scattering time* as the expectation value of t for the properly normalized probability $P(t)$. What is the mean scattering time if $n = 10^{22}$ molecules per cubic meter, $\langle v \rangle = 550$ m/s and, and $\sigma = \pi d^2$ for $d = 0.45$ nm?

Bonus, 5 pts. Use the expression of the mean scattering time τ to derive the *mean free path* as $\lambda \equiv \langle v \rangle \tau$. What is the mean free path using the same numerical inputs as part B?