08.128.140 Theoretische Physik 4: Statistische Physik
Theoretical Physics 4: Statistical Physics

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Homework set 4

Due November 20, 2023 by start of lecture.
Please note how long it took you to solve each problem.

4-1, 40 pts. *Mean and variance in binomial distribution, introduction to moments. For this problem, you may find it useful to read through Stowe, Chapter 3A; Kardar, Chapter 2.3; or Blundell and Blundell, Section 3.7.* We will consider a binomial distribution with probability p for some given condition. If you like, you can imagine this is the probability p for obtaining heads for an unfair coin, so we will keep p arbitrary but bounded by 0 and 1. Let $q = 1 - p$ be the complement of the probability p. Given the system has N total elements, then the probability that n elements satisfy the condition specified by probability p is given by the general binomial distribution,

$$P_N(n) = \frac{N!}{n!\,(N-n)!} p^n q^{N-n} \ . \tag{1}$$

If we take a large number of identically prepared systems (an ensemble), then the average number of elements per system that satisfy the criterion with probability p is

$$\langle n \rangle = pN \ . \tag{2}$$

As usual, we define the standard deviation as the square root of the average of the squared difference of elements from the mean value,

$$\sigma = \sqrt{\langle (n - \langle n \rangle)^2 \rangle} \ . \tag{3}$$

A, 10 pts. Show that $\sigma^2 = \langle n^2 \rangle - (\langle n \rangle)^2$ by expanding the expression and using the fact that $\langle n \rangle$ is a constant for the system.

B, 10 pts. Compute $\langle n^2 \rangle$ using the trick

$$n^2 p^n = \left(p \frac{\partial}{\partial p} \right)^2 p^n \ , \tag{4}$$

and show $\langle n^2 \rangle = (\langle n \rangle)^2 + Npq$. To use the trick, treat p and q as independent variables in $P_N(n)$ and evaluate afterwards subject to the constraint $p = 1 - q$. *Aside: this trick is central to the moment decomposition of probability distributions.*

C, 5 pts. Using the results from parts A and B, show $\frac{\sigma}{\langle n \rangle} = \sqrt{\frac{q}{Np}} \sim \frac{1}{\sqrt{N}}$.

D, 5 pts. Consider 600 air molecules in thermal equilibrium in an otherwise empty room. What is the average number of molecules $\langle n \rangle$ in the front third of the room, what is the standard deviation σ about this value, and what is the relative fluctuation $\sigma/\langle n \rangle$?

E, 5 pts. Repeat part D but now use 10^{26} as the number of air molecules.

F, 5 pts. How many air molecules would be needed to have a $\sigma/\langle n \rangle$ (relative fluctuation) of 10^{-10} in the number of air molecules in the front third of the room? *In other words, you can claim knowledge of the exact number of air molecules to one part in 10^{10} in the front third of the room in 68% ($\pm 1\sigma$) of cases, simply by virtue of the sheer number of molecules involved.*

4-2, 15 pts. *Boltzmann factor.* Assume a chemical reaction has an activation energy of $E_{\text{act}} \sim 1/2$ eV at room temperature, $T = 300$ K and assume the reaction probability follows a Boltzmann factor, $P \propto \exp(-E_{\text{act}}/(k_B T))$. Show that increasing the temperature to 310 K would lead to an approximate doubling in the reaction probability.

4-3, 30 pts. *Two-state system following the Boltzmann distribution.* Consider a two-state system that obeys the Boltzmann distribution where one state has energy 0 and the other has energy $\epsilon > 0$.

A, 5 pts. Construct the *partition function* of the system by summing the Boltzmann probability factors of each energy state.

B, 5 pts. Use the *partition function* to give a properly normalized expression for the probability of being the lower energy state $E = 0$ and the higher energy state $E = \epsilon$, given a fixed temperature T for the system.

C, 8 pts. What is the average energy of the system, $\langle E \rangle$?

D, 12 pts. Sketch the average energy in units of ϵ of the system as a function of $\epsilon/(k_B T)$ (in other words, plot in units of ϵ on the vertical axis and in units of $\epsilon/(k_B T)$ on the horizontal axis). What is the average energy of the system as $T \to \infty$ and $T \to 0$? What are the probabilities of each state in these two temperature limits?

4-4, 15 pts. *Open-ended writing prompt.* In your own words, after reading the corresponding sections on the second law of thermodynamics, Clausius's theorem, and the concept of entropy, and whatever else you find inspiring, write a paragraph (4-6 sentences) illustrating the importance of the second law of thermodynamics as it applies to the "real world." *You are free to choose any real-life example you find interesting. Suggestions include the need for renewable energy sources, the inevitable "heat death" of the universe, the application of entropy to information theory, the caloric intake needed for a daily routine, etc..* Full points will be awarded for accurate and appropriate use of physics keywords, and in particular, you should identify what cyclic process occurs in the real world and how input energy is required/consumed to power the cycle.