08.128.140 Theoretische Physik 4: Statistische Physik
Theoretical Physics 4: Statistical Physics

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Homework set 12

Due February 5, 2024 by start of lecture.
Please note how long it took you to solve each problem.

12-1, 20 pts. *Stefan-Boltzmann Law.* We will demonstrate in class that the radiation power law for a photon gas is σT^4, where σ is the Stefan-Boltzmann constant. For this problem, following Blundell and Blundell, Section 23.5, starting with an appropriate expression for the density of states for a photon gas in a cube of volume $V = L^3$, derive that

$$\sigma = \frac{\pi^2 k_B^4}{60 c^2 \hbar^3} = 5.67 \times 10^{-8} W/(m^2 K^4) \ . \tag{1}$$

You should assume that photons have two polarization states (to count the number of degrees of freedom) and also behave as simple quantum harmonic oscillators. Furthermore, you should neglect the energy density associated with the zero point energy of the harmonic oscillator, and you can use the intermediate result

$$\int_0^\infty \frac{x^3 dx}{e^x - 1} = \zeta(4)\Gamma(4) = \frac{\pi^4}{15} \ , \tag{2}$$

without proof.

12-2, 30 pts. *Realistic gases, the van der Waals equation.* A phenomenological description of a realistic gas (*i.e.* an equation of state that goes beyond the ideal gas law) is the van der Waals equation,

$$\left(p + \frac{a}{V_m^2}\right)(V_m - b) = RT \ , \tag{3}$$

where $V_m = V/n_{\text{moles}}$ is the volume occupied by 1 mole of gas, a is a constant that characterize the interactions between gas molecules and b is a constant that reflects the finite size of the gas molecules. *Aside: A full discussion of the van der Waals equation and its statistical mechanics origin is covered in Kardar, Chapter 5.1-5.4. A less rigorous discussion is presented in Blundell and Blundell, Section 26.1. For this problem, you are not required to read the Kardar discussion, but you are expected to read the Blundell and Blundell section.*

Derive the critical temperature T_c, pressure p_c, and volume V_c at the critical point of a van der Waals gas, characterized by solving for the point of inflection. Recall that the point of inflection (in a (p, V)-coordinate space) is defined by the simultaneous conditions that

$$\left(\frac{\partial p}{\partial V}\right)_T = 0 \ , \quad \left(\frac{\partial^2 p}{\partial V^2}\right)_T = 0 \ . \tag{4}$$

Also calculate the ratio $p_c V_c/(RT_c)$.

12-3, 25 pts. *Phase transitions, law of corresponding states.* For this problem, read through Blundell and Blundell, Section 26.4 and Section 28.7 as well as Kardar, Chapter 5.7 and 5.8. Write a paragraph that answers the following questions.

- How do you characterize a first order phase transition?
- How do you characterize a second order phase transition?
- Why do many materials with different microscopic properties show the same behavior near their critical points?

12-4, 25+10 pts. *Practice exam problem. Two-dimensional Fermi gas.* Consider an ideal Fermi gas of spin-1/2 particles of mass m in two spatial dimensions. The density of states as a function of energy density is given by $\mathcal{D}(\epsilon) = m/(\pi\hbar^2) = \text{const}$. The particle density $n \equiv \langle N\rangle/V$ as a function of temperature T and chemical potential μ is then given by

$$n = \int_0^\infty d\epsilon\, \mathcal{D}(\epsilon)\, \frac{1}{\exp(\beta(\epsilon - \mu)) + 1} \ . \tag{5}$$

A, 8 pts. For $T = 0$, determine the Fermi energy ϵ_F as a function of n. *Hint: See Section 30.2 of Blundell and Blundell. The Fermi energy is energy of the highest occupied state at a temperature of absolute zero.*

B, 8 pts. Calculate explicitly the "fugacity" $\exp(\beta\mu)$ as a function of the particle number density and the temperature. *Hint: Intermediate result is* $n = \frac{m}{\pi\hbar^2\beta}\ln(1 + \exp(\beta\mu))$.

C, 9 pts. Recall that isothermal compressibility κ_T is defined via

$$\kappa_T = -\frac{1}{V}\frac{\partial V}{\partial P} = \frac{1}{n^2}\frac{\partial n}{\partial \mu} \ . \tag{6}$$

Calculate the isothermal compressibility for the two-dimensional Fermi gas as a function of the particle number density n.

Bonus, 10 pts. Calculate the isothermal compressibility for a classical ideal gas as a function of particle number density n and compare to the result from part C in the highly dilute limit, $n \to 0$.