08.128.140 Theoretische Physik 4: Statistische Physik
Theoretical Physics 4: Statistical Physics

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Homework set 11

Due January 29, 2024 by start of lecture.
Please note how long it took you to solve each problem.

11-1, 45 pts. *Harmonic oscillator.* Given the quantum mechanical harmonic oscillator, we know the energy states of a single oscillator are $(n + \frac{1}{2})\hbar\omega$, with $n = 0, 1, 2, \ldots$. We will work in a canonical ensemble for this problem.

A, 10 pts. Calculate the single particle partition function $Z = \sum_n \exp[-\beta E_n]$ for the harmonic oscillator. Be sure to resum the geometric series.

B, 10pts. Calculate the expected internal energy as $E = -\frac{d \ln Z}{d\beta}$.

C, 10 pts. Calculate the heat capacity (at constant volume) as $C_V = \left(\frac{\partial E}{\partial T}\right)_V$.

D, 10 pts. Calculate the Helmholtz free energy $F = -k_B T \ln Z$ and entropy $S = (E - F)/T$.

E, 5 pts. Consider now N independent quantum oscillators, so the n from before becomes quantum occupation numbers $n_i = 0, 1, 2$ for the ith oscillator. The Hamiltonian is $H = \sum_{i=1}^{N} \hbar\omega(n_i + \frac{1}{2})$. What is the new partition function?

11-2, 30 pts. *Previous exam problem/Practice problem.* Consider a two-level system of N non-interacting particles, where each particle can have an energy $-\epsilon$ or $+\epsilon$.

A, 10 pts. Show the canonical partition function is $Z = 2^N \left(\cosh(\beta\epsilon)\right)^N$.

B, 10 pts. Calculate the free energy of the system and show that the entropy has the form

$$S = N k_B \left(\ln(2\cosh(\beta\epsilon)) - \beta\epsilon \tanh(\beta\epsilon)\right) \ . \tag{1}$$

C, 10 pts. Use the result from part (B) to calculate the average energy of the system.

11-3, 25 pts. *Blundell and Blundell, Problem 20.8.* The electron energy levels of an isolated hydrogen atom are given by $E_n = -R/n^2$, where $R = 13.6$ eV is the Rydberg energy. The degeneracy of each level is $2n^2$ (where the 2 arises becauses each electron has two spin degrees of freedom).

A, 10 pts. Show that

$$Z = \sum_{n=1}^{\infty} 2n^2 \exp\left(\frac{R}{n^2 k_B T}\right) \ . \tag{2}$$

B, 15 pts. Since the above expression diverges for $T \neq 0$ as $n \to \infty$, we can consider the case that the hydrogen atom is confined in a box of finite size and truncate the summation to a finite number of energy levels. If we approximate Z as dominated by the first two energy levels,

$$Z \approx \sum_{n=1}^{2} 2n^2 \exp\left(\frac{R}{n^2 k_B T}\right) , \tag{3}$$

estimate the mean energy of a hydrogen atom at $T = 300$ K.