08.128.140 Theoretische Physik 4: Statistische Physik
Theoretical Physics 4: Statistical Physics

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Homework set 10

Due January 22, 2024 by start of lecture.
Please note how long it took you to solve each problem.

10-1, 40 pts. *Heat capacity and occupation probabilities of two-level system.* Given the two-level system from Kardar, Section 4.3, we recall the energy of the system as a function of temperature,

$$E(T) = \frac{N\epsilon}{\exp(\frac{\epsilon}{k_B T}) + 1} \, , \tag{1}$$

where ϵ is the excited energy level.

A, 10 pts. Calculate the heat capacity $C = dE/dT$ of the system.

B, 5 pts. Sketch the heat capacity as a function of temperature.

C, 20 pts. Justify expressions for the occupation probabilities for the first site using the macrostate variable (*i.e. thermodynamic coordinate*) N_1, E, and T, starting from the unconditional probability for exciting the first impurity is

$$p(n_1) = \frac{\Omega(E - n_1\epsilon, N-1)}{\Omega(E, N)} \, . \tag{2}$$

D, 5 pts. Sketch the occupation probabilities $p(n_1 = 0)$ and $p(n_1 = 1)$ as functions of temperature from part (C).

10-2, 30 pts. *Phase space of ideal gas.* For the calculation of the ideal gas, we needed to perform an intermediate calculation the solid angle in a d-dimensional phase space (see Kardar, p. 105-106). Starting with

$$I_d \equiv \left(\int_{-\infty}^{\infty} dx \, e^{-x^2} \right)^d = \pi^{d/2} \, , \tag{3}$$

and given the integral representation of the factorial function as the Γ-function (see Eqs. (2.63) and (2.64) of Kardar),

$$\Gamma(N+1) \equiv N! = \int_0^{\infty} dx \, x^N \, e^{-x} \, , \tag{4}$$

show that the d-dimensional solid angle is

$$S_d = \frac{2\pi^{d/2}}{(d/2-1)!} \ , \tag{5}$$

and also justify that the expression for the total phase space for an ideal gas of N particles in a box of volume V with energy $[E-\Delta_E, E+\Delta_E]$ is

$$\Omega(E,V,N) = V^N \frac{2\pi^{3N/2}}{(3N/2-1)!}(2me)^{(3N-1)/2}\Delta_R \ , \tag{6}$$

where $\Delta_R = \sqrt{2m/E}\Delta E$.

10-3, 30+10 pts. *Gibbs paradox.* After reading Section 4.5 of Kardar (see also Section 21.5 of Blundell and Blundell), summarize and demonstrate the Gibbs paradox using the phase space expression from Problem 10-2 above. What is the required correction to this expression and what key property of equilibrium systems does it restore?

Bonus, 10 pts. In classical physics, distinguishability is not a meaningful concept since occupation numbers of identical phase space points are be arbitrarily high. What principle in quantum physics gives rise to the notion of (in)distinguishability? In other words, what is the fundamental reason that bosonic systems have symmetric wavefunctions under particle exchange while fermionic systems have asymmetric wavefunctions under particle exchange? This principle will serve as the basis for studying quantum statistical mechanics and lead to the Bose-Einstein and Fermi-Dirac distributions.