Lecture 4.
Supersymmetry
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From last time, we saw the first interactions between superfields.

Chiral superfields interact via superpotential W, which is a $\overset{\text{mass}}{\vee}$ dim. 3 polynomial in superfields.

$$\mathcal{L} = \int d^2\theta \; W(\Phi_i) + \text{h.c.}$$

For renormalizability, should have at most three powers of chiral superfield. (Remark: chiral superfields have mass dimension 1.)

Besides chiral superfield interactions, we will also want to study gauge interactions.

Shifman 49.8

Simplest construction is to promote gauge transformations to chiral superfields:

$$\phi(x) \to e^{i\alpha(x)}\phi(x), \quad \bar{\phi}(x) \to e^{-i\alpha(x)}\bar{\phi}(x),$$

$$A_\mu(x) \to A_\mu(x) + \partial_\mu \alpha(x)$$

For $\phi(x) \in \Phi(x_L, \theta)$, then $\alpha(x) \to \Lambda_L(x_L, \theta)$

but then $\bar{\phi}(x) \in \bar{\Phi}(x_R, \bar{\theta})$ and $-\alpha(x) \to -\bar{\Lambda}_R(x_R, \bar{\theta})$

We then get

$$\Phi_L(x_L, \theta) \to e^{i\Lambda_L(x_L, \theta)}\,\Phi_L(x_L, \theta)$$

$$\bar{\Phi}_R(x_R, \bar{\theta}) \to e^{-i\bar{\Lambda}_R(x_R, \theta)}\,\bar{\Phi}_R(x_R, \theta)$$

(*) and $V(x, \theta, \bar{\theta}) \to V(x, \theta, \bar{\theta}) - i\left[\Lambda_L(x_L, \theta) - \bar{\Lambda}_R(x_R, \bar{\theta})\right]$

Gauge invariant term is then $\boxed{\mathcal{L} = \int d^4\theta \; \bar{\Phi}_R \, e^{V} \, \Phi_L}$

By a parametrization choice of $\Lambda_L(x_L,\theta)$, $\bar{\Lambda}_R(x_R,\bar{\theta})$, we modify the components of $V(x,\theta,\bar{\theta})$ under the gauge transformation in ☆.

In particular, for $\Lambda_L(x_L,\theta) = \varphi + \sqrt{2}\,\theta\eta + \theta^2 F$,

then $C \to C - i(\varphi - \bar{\varphi})$, $\chi \to \chi - \sqrt{2}\eta$, $M \to M - \sqrt{2}F$,

$v_{\alpha\dot{\beta}} \to v_{\alpha\dot{\beta}} + \frac{1}{2}\partial_{\alpha\dot{\beta}}(\varphi + \bar{\varphi})$, $\lambda \to \lambda$, $D \to D$.

Choosing $C - i(\varphi - \bar{\varphi}) = 0$ and also the $\theta^2\bar{\theta}^2$ component of $V = \theta^2\bar{\theta}^2\left(D - \frac{1}{4}\partial^2 C\right) \to \theta^2\bar{\theta}^2 D$.

Can eliminate C, χ, M components $\Rightarrow$ Wess-Zumino gauge.

Then $$V(x,\theta,\bar{\theta}) = -2\theta^\alpha\bar{\theta}^{\dot{\alpha}}A_{\alpha\dot{\alpha}} - 2i\bar{\theta}^2(\theta\lambda) + 2i\theta^2(\bar{\theta}\bar{\lambda}) + \theta^2\bar{\theta}^2 D.$$

Recall V is real, $V = V^\dagger$.

For non-Abelian vector superfields, generalize Λ to a matrix-valued transformation, then V becomes matrix-valued, and then the corresponding group structure is carried by the commutation rules carried by the matrices (in other words, by the Lie algebra).

Hence $V(x,\theta,\bar{\theta}) \equiv V^a T^a$

$\Lambda_L(x_L,\theta) = \Lambda_L^a T^a$, $\bar{\Lambda}_R(x_R,\bar{\theta}) = \bar{\Lambda}^a T^a$

for T^a as the usual Hermitian generators of the group.

We now need the field strength tensor.

This is defined by

$$W_\alpha = \frac{1}{8}\bar{D}^2 D_\alpha V = i(\lambda_\alpha + i\theta_\alpha D - \theta^\beta F_{\alpha\beta} - i\theta^2\partial_{\alpha\dot{\alpha}}\bar{\lambda}^{\dot{\alpha}})$$

for $U(1)$ field strengths.

$$F_{\alpha\beta} \equiv -\frac{1}{2}F_{\mu\nu}(\sigma^\mu)_{\alpha\dot{\gamma}}(\bar{\sigma}^\nu)_\beta{}^{\dot{\gamma}}$$

Note that W_α is chiral, since
$$\bar{D}_{\dot\alpha} W_\alpha = 0.$$
Motivation for this definition:
Recall
$$\{D_\alpha, D_\beta\} = \{\bar{D}_{\dot\alpha}, \bar{D}_{\dot\beta}\} = 0$$
$$\{D_\alpha, \bar{D}_{\dot\beta}\} = 2i\,\partial_{\alpha\dot\beta}$$

So action of two D_α is ≈ a normal vector derivative.
The Lagrangian term for the field strength is then
$$\frac{1}{4e^2}\int d^2\theta W^2 + h.c.$$
with e the $U(1)$ gauge coupling and $W^2 = W^\alpha W_\alpha$.

Calculating W^2, we get
$$W^2(x_L, \theta) = -\lambda^2 - 2i(\lambda\theta)D + 2\lambda^\alpha F_{\alpha\beta}\theta^\beta + \theta^2\left(D^2 - \frac{1}{2}F^{\alpha\beta}F_{\alpha\beta}\right) + 2i\theta^2\bar{\lambda}_{\dot\alpha}\partial^{\dot\alpha\alpha}\lambda_\alpha$$
So the $\int d^2\theta$ will extract the expected $F^{\alpha\beta}F_{\alpha\beta}$.
Note W has mass dimension $3/2$.

For non-Abelian, we make V, W_α^a as matrix-valued

Shifman p.473
$$V(x, \theta, \bar\theta) = V^a T^a$$
$$W_\alpha = \frac{1}{8}\bar{D}^2 e^{-V}(D_\alpha e^V)$$
$$W^2(x_L, \theta) = -\lambda^2 - 2i(\lambda\theta)D + 2\lambda^\alpha G_{\alpha\beta}\theta^\beta + \theta^2\left(D^2 - \frac{1}{2}G^{\alpha\beta}G_{\alpha\beta}\right) + 2i\theta^2\bar{\lambda}_{\dot\alpha}D^{\dot\alpha\alpha}\lambda_\alpha$$
where the usual non-Abelian group structure is encoded by the Lie algebra governing the T^a matrices:
$$[T^a, T^b] = i f^{abc} T^c$$
The gauge transformations from before also become matrix-valued. $\Lambda_L(x_L, \theta) \equiv \Lambda_L^a T^a$, $\bar{\Lambda}_R(x_R, \bar\theta) = \bar{\Lambda}_R^a T^a$

Holomorphic couplings:

In both the superpotential Lagrangian and the fieldstrength tensor, we construct the Lagrangian by integrating over the superfields on the chiral subspace spanned by (x_L, θ) or $(x_R, \bar{\theta})$;

$$\mathcal{L} \supset \left(\int d^2\theta \; W(\Phi_i) + \frac{1}{4e^2} W^\alpha W_\alpha \right) + h.c.$$

where $W(\Phi_i) = \frac{m}{2}\Phi^2 + \frac{\lambda}{3}\Phi^3$ for the Wess-Zumino model

The coupling constants are complex parameters, but they never show up as, e.g. $|m|^2$, but always as analytic fcns. of couplings $\equiv$ holomorphic.

Recall from complex analysis: $z = a + ib$,

z^2 is analytic ($\mathbb{C}$ differentiable)

while $|z|^2 = z z^*$ is not analytic everywhere.

We can prove holomorphy in couplings by a spurion analysis, where $m \to M(x_L, \theta) = m + \ldots$ is promoted to a chiral superfield. Then the chiral subspace extends for the entire integrand, and we will never have M^* appearing in the superpotential.

Aside: the holomorphic gauge coupling has a real piece corresponding to the usual gauge coupling + the imaginary piece is the θ-angle.

$$\frac{1}{e^2} \equiv \frac{1}{\tilde{e}^2} - i\frac{\theta}{8\pi^2} \qquad [\text{arb. norm.}]$$

$$\mathcal{L} \supset \frac{-1}{4\tilde{e}^2} F^{\mu\nu}F_{\mu\nu} + \frac{\theta}{32\pi^2} F_{\mu\nu}\tilde{F}^{\mu\nu}$$

$$+ \frac{1}{\tilde{e}^2}\bar{\lambda}_{\dot{\alpha}}\, i\partial^{\dot{\alpha}\alpha}\lambda_\alpha + \frac{1}{2\hat{e}^2}D^2$$

Introducing R-symmetry:

Completely global $U(1)$ symmetry generators should ~~comp~~ commute with all generators of super-Poincaré algebra. But there is a subcategory of $U(1)$ symmetry generators that do not commute with $Q_\alpha, \bar{Q}_{\dot\alpha}$:

$$[R, Q_\alpha] = -Q_\alpha, \qquad [R, \bar{Q}_{\dot\alpha}] = +\bar{Q}_{\dot\alpha}$$

This is an extra $U(1)$ symmetry called $U(1)_R$, which basically charges the superspace coordinates θ & $\bar{\theta}$ as $+1$ and -1, respectively. We conventionally give the same R-charge for superfield Φ and its lowest component ϕ. Since the Lagrangian is obtained by $\left(\int d^2\theta\, W + h.c.\right)$, the superpotential W carries R-charge $+2$ (since $d\theta$ has R-charge -1).

Note: R-symmetry is not equivalent to R-parity, but R-parity will generally arise from R-symmetry.
Note 2: Any global $U(1)$ combined with $U(1)_R$ is another R-symmetry.

Non-renormalization of the superpotential:

The combination of holomorphy and R-symmetry is enough to prove the non-renormalization of the superpotential. More precisely, new superpotential terms are not generated that violate holomorphy or R-symmetry.

As an example,

$$W_{\text{Wess-Zumino}} = \frac{m}{2}\Phi^2 + \frac{\lambda}{3}\Phi^3$$

For R-symmetry of W to be 2, we need $R(\Phi) = \frac{2}{3}$.
and $R(\lambda) = 0$. Then $R(m) = \frac{2}{3}$.
We also have a separate global $U(1)$ symmetry with
$Q(\Phi) = 1$, $Q(m) = -2$, $Q(\lambda) = -3$.
As is usual in spurion analysis, we promote couplings to vevs of background fields. The spurious symmetry structure dictates the renormalization group flow.

Terning, p.134
Shifman, p.448

Now, we build the most general superpotential from

$$W_{eff} = m\Phi^2 f\left(\frac{\lambda\Phi}{m}\right) \quad \text{for } f\left(\frac{\lambda\Phi}{m}\right) \text{ as power series.}$$

Note $R\left(\frac{\lambda\Phi}{m}\right) = 0$ + $Q\left(\frac{\lambda\Phi}{m}\right) = 0$.

$$W_{eff} = c_n \lambda^n \Phi^{n+2} m^{1-n}$$

To smoothly reach the massless limit, $m \to 0$, $n \leq 1$ and for the weak coupling limit, $\lambda \to 0$, $n \geq 0$, and we get

$$W_{eff} = \frac{m}{2}\Phi^2 + \frac{\lambda}{3}\Phi^3 = W_{tree}.$$

There is renormalization in the kinetic term = wavefunction renormalization.

$$\int d^4\theta\, \bar{\Phi}\Phi \to \int d^4\theta\, Z\, \bar{\Phi}\Phi$$

$$+ \quad m_r = \frac{m}{Z}, \quad \lambda_r = \frac{\lambda}{Z^{3/2}}$$

Note $\frac{m_r^3}{\lambda_r^2} = \frac{m^3}{\lambda^2}$ has no Z dependence + is not renormalized.
(Corresponds to domain wall tension.)