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Metric is always $(+,-,-,-)$

Introduction + Motivation

Distinction between studying "real" phenomena + "aesthetic"/theory problems.

Real = DM, baryogenesis, quantum gravity

Aesthetic = "Fine-tuning" hierarchy problem, θ-QCD, cosmological constant

Similar to distinction between bottom-up + top-down approaches to model-building.

Bottom-up: Add particles for ν-mass, DM, flavor puzzles in $g-2$ or R_{K/K^*} anomalies.

Parametrize NP effects by EFT, agnostic about possible coupling patterns in UV physics!

Top-down: Impose new symmetry; spurion analysis can be very useful!

Will adopt top-down perspective: Add a new symmetry to $\mathcal{L}$.

Should already know how to analyze global symmetries + gauge symmetries.

Recall that these symmetries (if non-anomalous), leave action invariant, leading to conserved Noether currents + charges commute with Hamiltonian, ~~and~~ making the charges good quantum numbers. ~~to factorize the Hilbert space.~~

Given that you can readily build more complicated QFTs from adding more + more gauge + global symmetries with more dofs, what about geometric extensions?

Coleman-Mandula theorem: Phys. Rev. 159, 1251 (1967)

Shifman, Chap. 46, p. 413.

Following Witten: (Intro to SUSY, ed. Zichichi, The Unity of the Fundamental Interactions, 1983)

For theories with nontrivial S matrix, no geometric extensions of Poincaré algebra are possible.
Can only have have 4 P_μ (energy-momentum operator) and six Lorentz transformations $M_{\mu\nu}$. Rest must be scalars such as EM charge.

~~Loophole~~

Wess + Bagger, p. 4

Explicitly, Coleman-Mandula assumes:

1.) S-matrix is based on local, relativistic QFT in 4D
2.) There are only finite # of diff. particles associated with one-particle states of a given mass.
3.) There is an energy gap between the vacuum + one particle states.

Then the most general Lie algebra of symmetries of S-matrix contains $\underbrace{P_m}_{\text{energy-momentum op.}}$, Lorentz rotation generator M_{mn}, + finite # of Lorentz scalar operators B_ℓ. Further, B_ℓ must belong to Lie algebra of a compact Lie group.

Golfand + Likhtman (1970)

Loophole: Generalize Lie algebra to graded Lie algebra, which allows anticommutators + commutators. Can then introduce spinorial generators with anticommutation relations.

Recall Lorentz boosts, rotations + translations form Poincaré group. Lorentz group is $SO(\overset{SL(2,\mathbb{C})}{\overset{\shortparallel}{1},3})$, Poincaré is $\mathbb{R}^{1,3} \times SO(1,3)$

Peskin, 3.1

$$\vec{J} = \vec{x} \times \vec{p} = \vec{x} \times (-i\vec{\nabla})$$

generalize to $J^{\mu\nu} = i(x^\mu \partial^\nu - x^\nu \partial^\mu)$, antisymmetric tensor

Commutator is

$$[J^{\mu\nu}, J^{\rho\sigma}] = i(g^{\nu\rho} J^{\mu\sigma} - g^{\mu\rho} J^{\nu\sigma} - g^{\nu\sigma} J^{\mu\rho} + g^{\mu\sigma} J^{\nu\rho})$$

Can define generators of rotations + boosts as

$$L^i = \frac{1}{2} \epsilon^{ijk} J^{jk}$$

$$K^i = J^{0i} \qquad \text{for } i,j,k = 1,2,3.$$

Infinitesimal Lorentz transformation is

$$\Phi \to (1 - i\vec{\theta}\cdot\vec{L} - i\vec{\beta}\cdot\vec{K})\,\Phi$$

Finite dim. reps are those with integer or half-integer values for angular momentum.

Poincaré adds momentum generators to Lorentz.

$$[p^\mu, p^\nu] = 0$$

$$[J^{\mu\nu}, p^\rho] = i(-g^{\mu\rho} p^\nu + g^{\nu\rho} p^\mu)$$

Correspond to conservation of energy + momentum + angular momentum.

Shifman 47.2

(In Weyl representation) Introduce 4 new <u>fermionic</u> ~~conser~~ supercharges, Q_α, $\bar{Q}^{\dot\alpha}$ where $\alpha, \dot\alpha = 1,2$.

$$[p_\mu, Q_\alpha] = [p_\mu, \bar{Q}^{\dot\alpha}] = 0$$

$$[J^{\mu\nu}, Q_\alpha] = i(\sigma^{\mu\nu})_\alpha^{\ \beta} Q_\beta$$

$$[J^{\mu\nu}, \bar{Q}^{\dot\alpha}] = i(\bar\sigma^{\mu\nu})^{\dot\alpha}_{\ \dot\beta} \bar{Q}^{\dot\beta}$$

Since Q_α, $\bar{Q}^{\dot\alpha}$ transform as Weyl spinors:

$$\sigma^{\mu\nu} \equiv \frac{1}{4}(\sigma^\mu \bar\sigma^\nu - \sigma^\nu \bar\sigma^\mu) = \left(-\frac{i}{2}\vec\sigma, \frac{i}{2}\vec\sigma\right)$$

$$\bar\sigma^{\mu\nu} \equiv \frac{1}{4}(\bar\sigma^\mu \sigma^\nu - \bar\sigma^\nu \sigma^\mu) = \left(\frac{i}{2}\vec\sigma, \frac{i}{2}\vec\sigma\right)$$

$F^{\mu\nu} = (-\vec{E}, \ldots$

$F_{\mu\nu} = (\vec{E}, \ldots$

Key point: Need anticommutators $\{Q_\alpha, \bar{Q}_{\dot\beta}\}$ and $\{Q_\alpha, Q_\beta\}$ to close the algebra.

$$\{Q_\alpha, \bar{Q}_{\dot\beta}\} = 2 p_\mu (\sigma^\mu)_{\alpha\dot\beta} = 2 P_{\alpha\dot\beta}$$

And choose $\{Q_\alpha, Q_\beta\} = \{\bar{Q}_{\dot\alpha}, \bar{Q}_{\dot\beta}\} = 0$

Alternative is $\{Q_\alpha, Q_\beta\} = C_{\alpha\beta}$, where $C_{\alpha\beta}$ is a triplet of central charges which commute w/ all other generators, like topological currents + charges.

Comments:

1.) Can also have $\mathcal{N}=2$, $\mathcal{N}=4$ SUSY in 4D. Put
$$\{Q_\alpha{}^A, \bar{Q}_{\dot\beta B}\} = 2\sigma^m_{\alpha\dot\beta} P_m \delta^A{}_B \quad \text{for } A,B = 1,..,\mathcal{N}.$$

2.) Coleman + Mandula theorem briefly:

Only symmetry of S-matrix that includes Poincaré is product of Poincaré and internal symmetry group.

Suppose false: Additional symmetry generators transform as Lorentz tensors would ~~ce~~ over-constrain S-matrix. Recall in 2-2 scattering, S-matrix is fcn. only of one variable, scattering angle. If tensor current is conserved, then angle is no ~~...~~ longer continuous + S-matrix is not ~~...~~ analytic.

SUSY evades Coleman + Mandula with anticommutators + spinor generators.

Terning, p. 5 3) SUSY algebra is invariant under multiplication of Q_α by a phase, so there is one linear combination of U(1) charges, "R-charge" that does not commute with Q + $Q^\dagger$:

$$[Q_\alpha, R] = Q_\alpha, \quad [Q^\dagger_{\dot\alpha}, R] = -Q^\dagger_{\dot\alpha}.$$

This is R-symmetry (later, gives origin of R-parity)

The SUSY algebra extends the Poincaré algebra to a super-Poincaré algebra. Fundamentally distinct from other kinds of symmetries that you can consider, like gauge + global symmetries.

Shifman p. 422 In non-SUSY field theories, we had fields of spin 0, ½, + 1 that depend locally ~~on~~ spacetime x^μ.

Just as the Energy-momentum tensor generates translations in 4D spacetime, anticommuting supercharges should generate "super" translations in an anticommuting space.

Introduce $\theta^\alpha, \bar\theta^{\dot\alpha}$ as four anti-commuting "fermionic" dimensions of superspace. Recall $\alpha, \dot\alpha = 1,2$.

Group element of $N=1$ superalgebra is then

$$G(x^\mu, \theta, \bar\theta) = \exp(i(\theta Q + \bar\theta\bar Q - x^\mu P_\mu))$$

with Grassmann variables θ^α, $\bar\theta^{\dot\beta} \equiv (\theta^\beta)^*$

Recall Grassmann variables (from fermionic path integral approach to QFT) anticommute:

Peskin, p. 302

$$\{\theta^\alpha, \theta^\beta\} = \{\bar\theta^{\dot\alpha}, \bar\theta^{\dot\beta}\} = \{\theta^\alpha, \bar\theta^{\dot\beta}\} = 0$$

$$\left\{\frac{\partial}{\partial\theta^\alpha}, \frac{\partial}{\partial\theta^\beta}\right\} = \left\{\frac{\partial}{\partial\bar\theta^{\dot\alpha}}, \frac{\partial}{\partial\bar\theta^{\dot\beta}}\right\} = \left\{\frac{\partial}{\partial\theta^\alpha}, \frac{\partial}{\partial\bar\theta^{\dot\beta}}\right\} = 0$$

What are the reps. in superspace?

Will see that under SUSY transformations, scalars transform into fermions & vice-versa.

Consider how the group elements in $G(x^\mu, \theta, \bar\theta)$ act on $G(a^\mu, \epsilon, \bar\epsilon)$.

$$G(x^\mu, \theta, \bar\theta)\, G(a^\mu, \epsilon, \bar\epsilon)$$

Exercise: $$= G(x^\mu + a^\mu + i\epsilon\sigma^\mu\bar\theta - i\theta\sigma^\mu\bar\epsilon,\ \theta+\epsilon,\ \bar\theta+\bar\epsilon)$$

which follows from Hausdorff formula

~~Exercise~~ $$e^A e^B = \exp\left(A + B + \tfrac{1}{2}[A,B] + \ldots\right)$$

We identify $x^\mu \to x^\mu + \delta x^\mu$

with $$\delta x^\mu \equiv \delta\left(\tfrac{1}{2} x_{\alpha\dot\beta} (\bar\sigma^\mu)^{\dot\beta\alpha}\right)$$
$$= \tfrac{1}{2}(\delta x_{\alpha\dot\beta})(\bar\sigma^\mu)^{\dot\beta\alpha}$$

with $$\delta x_{\alpha\dot\beta} = -2i\theta_\alpha\bar\epsilon_{\dot\beta} - 2i\bar\theta_{\dot\beta}\epsilon_\alpha$$

and $$\delta\theta^\alpha = \epsilon^\alpha, \quad \delta\bar\theta^{\dot\alpha} = \bar\epsilon^{\dot\alpha}$$

Aside: Weyl spinors and 2-component spinor notation

$$\psi = \begin{pmatrix}\xi\\ \eta\end{pmatrix} \qquad P_L\psi = \xi \quad \text{LH-spinor}$$
$$P_R\psi = \eta \quad \text{RH-spinor}$$

Peskin, p.44.

$$\xi_L \to \exp\left(1 - i\vec\alpha\cdot\frac{\vec\sigma}{2} - \vec\beta\cdot\frac{\vec\sigma}{2}\right)\xi_L$$
$$\eta_R \to \exp\left(1 - i\vec\alpha\cdot\frac{\vec\sigma}{2} + \vec\beta\cdot\frac{\vec\sigma}{2}\right)\eta_R$$

Infinitesimal rotations $\vec\alpha$, boosts $\vec\beta$.

Recall $\sigma^2\vec\sigma^* = -\vec\sigma\sigma^2$ & so $\sigma^2(\xi_L^*)$ transforms as a RH-spinor.

Notation: undotted α, β are spinor indices for LH Weyl spinors.

$\dot\alpha, \dot\beta$ are spinor indices for RH Weyl spinors.

To keep track of upper + lower indices, use

$$\epsilon^{\alpha\beta} = -\epsilon^{\alpha\beta}, \quad \epsilon^{12} = 1 = -\epsilon_{12}$$

$$\epsilon^{\dot\alpha\dot\beta} = -\epsilon^{\dot\beta\dot\alpha}, \quad \epsilon^{\dot 1\dot 2} = 1 = -\epsilon_{\dot 1\dot 2}$$

Contracting spinors:

$$\eta\chi \equiv \eta^\alpha \chi_\alpha, \quad \bar\eta\bar\chi \equiv \bar\eta_{\dot\alpha}\bar\chi^{\dot\alpha}$$

where ~~[illegible]~~. Can show

$$\bar\chi_{\dot\alpha} \equiv (\chi_\alpha)^*, \quad \bar\eta^{\dot\alpha} \equiv (\eta^\alpha)^*$$

$$(\eta\chi)^\dagger = (\eta^\alpha\chi_\alpha)^\dagger = (\chi_\alpha)^*(\eta^\alpha)^* = \bar\chi\bar\eta.$$

We also have fermion bilinears

$$\chi^\alpha\chi^\beta = -\tfrac{1}{2}\epsilon^{\alpha\beta}\chi^2, \quad \chi_\alpha\chi_\beta = +\tfrac{1}{2}\epsilon_{\alpha\beta}\chi^2$$

$$\bar\chi^{\dot\alpha}\bar\chi^{\dot\beta} = \tfrac{1}{2}\epsilon^{\dot\alpha\dot\beta}\bar\chi^2, \quad \bar\chi_{\dot\alpha}\bar\chi_{\dot\beta} = -\tfrac{1}{2}\epsilon_{\dot\alpha\dot\beta}\bar\chi^2.$$

To extend to vectors, $(\tfrac{1}{2}, \tfrac{1}{2})$ rep. of Lorentz, use $(\sigma^\mu)_{\alpha\dot\beta} = \{1, \vec\sigma\}_{\alpha\dot\beta}$ for convolution.

$$A_\mu \equiv (A^0, -\vec A)$$

$$A^\mu = \tfrac{1}{2} A_{\alpha\dot\beta}(\bar\sigma^\mu)^{\dot\beta\alpha}, \quad A_{\alpha\dot\beta} = A_\mu(\sigma^\mu)_{\alpha\dot\beta}$$

Consider $(\sigma^\mu)_{\alpha\dot\beta}$ RH + $(\bar\sigma^\mu)^{\dot\beta\alpha}$ LH.

Recall Dirac equation:

$$\bar\Psi\, i\not\partial\,\Psi - m\bar\Psi\Psi = 0$$

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar\sigma^\mu & 0 \end{pmatrix}$$

For free, $\not\partial \to \not\partial \Rightarrow$

$$\begin{pmatrix} -m & i\sigma\cdot\partial \\ i\bar\sigma\cdot\partial & -m \end{pmatrix}\begin{pmatrix} \Psi_L \\ \Psi_R \end{pmatrix} = 0$$

or

$$i\sigma\cdot\partial\,\Psi_R = m\Psi_L$$

$$i\bar\sigma\cdot\partial\,\Psi_L = m\Psi_R$$

Can extend this distinction to superspace.

Two invariant subspaces:

$$\{x_L^\mu, \theta^\alpha\}, \quad \delta\theta^\alpha = \epsilon^\alpha, \quad \delta(x_L)_{\alpha\dot\alpha} = -4i\,\theta_\alpha \bar\epsilon_{\dot\alpha}$$

$$\{x_R^\mu, \bar\theta^{\dot\alpha}\}, \quad \delta\bar\theta^{\dot\alpha} = \bar\epsilon^{\dot\alpha}, \quad \delta(x_R)_{\alpha\dot\alpha} = -4i\,\bar\theta_{\dot\alpha}\epsilon_\alpha$$

with $(x_L)_{\alpha\dot\alpha} = x_{\alpha\dot\alpha} - 2i\,\theta_\alpha\bar\theta_{\dot\alpha}$

$(x_R)_{\alpha\dot\alpha} = x_{\alpha\dot\alpha} + 2i\,\theta_\alpha\bar\theta_{\dot\alpha}$

Will also use vectorial notation

$$x_L^\mu = x^\mu - i\theta^\alpha(\sigma^\mu)_{\alpha\dot\alpha}\bar\theta^{\dot\alpha},$$

$$x_R^\mu = x^\mu + i\theta^\alpha(\sigma^\mu)_{\alpha\dot\alpha}\bar\theta^{\dot\alpha}$$

In SUSY, we have superfields as ~~coords~~ fcns. on superspace coordinates. Expanding superfield in powers of supervariables θ^α + $\bar\theta^{\dot\alpha}$, we get the regular fields as ~~coordinat~~ components of the superfield.

Remark: Expansion is finite since Taylor expansion over Grassmann variables is finite. Highest term is $\theta^2\bar\theta^2 \equiv \theta^\alpha\theta_\alpha\bar\theta_{\dot\alpha}\bar\theta^{\dot\alpha}$

Most general superfield is

$$S(x,\theta,\bar\theta) = \phi + \theta\psi + \bar\theta\bar\chi + \theta^2 F + \bar\theta^2 G + \theta^\alpha A_{\alpha\dot\beta}\bar\theta^{\dot\beta} + \theta^2(\bar\theta\bar\lambda) + \bar\theta^2(\theta\rho) + \theta^2\bar\theta^2 D,$$ with

$\phi, \psi, \bar\chi, \ldots D$ as component fields depending only on x^μ.

Next time: ~~Reducible reps.~~

Reps. of SUSY algebra based on $S(x,\theta,\bar\theta)$.