QFT 3.
The Standard Model and EW Theory.
Special topic lecture: Chiral Lagrangian
Felix Yu
June 13, 2024.

In this lecture, we will discuss quark-hadron duality, as dictated by the "chiral Lagrangian." One key aspect of the chiral Lagrangian is ability to dictate interactions of composite objects after chiral symmetry breaking and not just give a taxonomy that accounts for degrees of freedom.

Recall the QCD Lagrangian. We have $SU(3)_c$ gauge symmetry with a vector-like representation of quarks. The fact that the quarks are vector-like (i.e. LH and RH quarks have the same interaction with the gluon field) is key, and it is manifest when analyzing the fermion mass basis interactions of the SM. Namely, after EWSB, the quarks have $SU(3)_c \times U(1)_{em}$ interactions with diagonal masses + Yukawa couplings, and a mix of vector + axial-vector couplings to the $W^{+/-}$ and Z bosons. Neglecting the $SU(2)_L$ gauge bosons, Yukawas, we have

$$\mathcal{L}_{QCD} \supset -\frac{1}{4} G^a_{\mu\nu} G^{a\mu\nu} + \bar{u}\, i\not{D} u + \bar{d}\, i\not{D} d$$
$$+ \bar{s}\, i\not{D} s + (\bar{c}\, i\not{D} c + \bar{b}\, i\not{D} b + \bar{t}\, i\not{D} t)$$
$$- m_u \bar{u} u - m_d \bar{d} d + m_s \bar{s} s \;(- m_c, m_b, m_t)$$
$$+ (\text{Yukawas, EW interactions})$$

where $\not{D} = \begin{cases} \gamma^\mu\left(\partial_\mu - i g_s t^a G_\mu^a - ie\frac{2}{3}\cdot A_\mu\right) & \text{for up} \\ \gamma^\mu\left(\partial_\mu - i g_s t^a G_\mu^a - ie\left(-\frac{1}{3}\right) A_\mu\right) & \text{for down} \end{cases}$

is the covariant derivative for up-type and down-type quarks.
Note the Lagrangian does not have (+h.c.) because all of the fermion bilinears are already Hermitian.

Now, we also know QCD in the SM is asymptotically free. The β-fcn.,

$$\beta(g) = -\frac{g^3}{16\pi^2}\left(\frac{4}{3}C_2(G) - \frac{4}{3}n_f C(r_f)\right)$$
$$= -\frac{g^3}{16\pi^2}\left(11 - \frac{2}{3}N_F\right)$$

runs from small values in the UV to large values in the IR, with a scale of strong coupling Λ_{QCD} ~ 200 MeV. As a consequence, the quarks become bound states of mesons and baryons when their energies are comparable to Λ_{QCD}, and these bound states are composite objects that are color singlets: mesons are $\bar{q}q$ (anti-fundamental $\otimes$ fundamental) & baryons are qqq (ϵ^{ijk} fundamental $\otimes$ fund. $\otimes$ fund.)

How do these bound states interact and what is their mass spectrum? In particular, what remains as "force carriers" for the strong interaction after confinement. Intuitively, we know that composite objects that are null charge in total can still interact via the corresponding current, albeit as dipole or higher multipole interactions. (Consider EM interactions of atoms, for example.)

The way to address this question is to transform the QCD Lagrangian into a new, effective description that uses different degrees of freedom whose dynamics are inherited by the ~~elemen~~ elementary description.
We recall the masses of the quarks from PDG:

$$m_u = 2.16 \pm 0.07 \text{ MeV}$$
$$m_d = 4.70 \pm 0.07 \text{ MeV}$$
$$m_s = 93.5 \pm 0.8 \text{ MeV}$$
$$m_c = 1.2730 \pm 0.0046 \text{ GeV}$$
$$m_b = 4.183 \pm 0.007 \text{ GeV}$$
$$m_t = 172.4 \pm 0.7 \text{ GeV (pole)}$$

We see that the up, down, and strange quarks all have masses below the nominal Λ_{QCD} scale, so we can consider these masses as subleading corrections to the dynamics governed by the quark condensate.

The starting point of the chiral Lagrangian is the assumption that QCD at strong coupling causes a chiral condensate of quarks to get a vacuum expectation value.

$$\langle q_{La} q^{\dagger}_{Rb} \rangle = \tfrac{1}{2}\Delta\, \delta_{ab}$$

a, b indices running from 1 to N_F.

cf.
Weinberg, Vol 2, Chap. 19

Pich, hep-ph/9502366

Scherer, hep-ph/0210398

Donoghue, Golowich, Holstein - Dynamics of the SM

With this starting point for the QCD vacuum, we have an ultraviolet description of a QCD-like theory built from N_F quarks transforming as vector-like fundamental reps. under QCD.

In this vein, the Lagrangian of the elementary particles is approximately:

$$\mathcal{L} = \sum_{j=1}^{N_F} \bar{q}_j \, i \not{D} q_j - \frac{1}{4} G^a_{\mu\nu} G^{a\mu\nu}$$

Clearly, the elementary particles have a global symmetry of $SU(N_F)_L \times SU(N_F)_R \times U(1)_V \times U(1)_A$, because the quarks, as 4-component fields, are built from LH and RH Weyl fermions, which can transform into each other via $U(N_F)_L \times U(N_F)_R = SU(N_F)_L \times SU(N_F)_R \times U(1)_V \times U(1)_A$ transformations. Specifically, $N_F = 3$ for the SM QCD Lagrangian.

Correspondingly, we can map out the moduli space by acting the global symmetry generators on the vacuum state. If these global symmetries are exact, then the moduli space is an exact degeneracy of vacuum energies which corresponds to Goldstone field directions. Even if these global symmetries are explicitly broken, as long as the breaking of these symmetries are dimensionful (i.e. soft-breaking) and not dimensionless (i.e. hard breaking), we can dictate the interactions & spectrum of Goldstone bosons by a consistent symmetry-breaking approach.

Under global $SU(N_F)_L \times SU(N_F)_R$, the elementary description of the theory, i.e. quarks, transform via

$$\begin{pmatrix} u \\ d \\ s \end{pmatrix} \to \begin{pmatrix} u' \\ d' \\ s' \end{pmatrix} = \exp\left[i \sum_a \left(\theta^V_a \lambda_a + \theta^A_a \lambda_a \gamma^5 \right) \right] \begin{pmatrix} u \\ d \\ s \end{pmatrix}$$

where λ_a are the Gell-Mann generators of $SU(3)$.

Since our vacuum state is

$$\langle q_{La} q^\dagger_{Rb} \rangle = \frac{1}{2} \Delta \, \delta_{ab},$$

we can recognize that the vacuum state leaves the vector symmetry subgroup $SU(3)_V \in SU(3)_L \times SU(3)_R$ unbroken [you can imagine writing P_L + P_R transformations of the quark fields and noting that $q_L q_R^\dagger$ is invariant for LH = RH symmetry parameters]. Consequently, the vacuum state spontaneously breaks the axial $SU(3)$ flavor symmetry. Moreover, the axial symmetry transformation encodes the physical Goldstones of the theory.

Following Weinberg, these Goldstones are the parameters ξ_a that determine the right coset of the breaking pattern $SU(3) \times SU(3) / SU(3)$, and we can write it as a matrix field $\gamma(\xi) = \exp\left[-i \gamma^5 \sum_a \xi_a \lambda_a \right]$.

(Minus sign for ~~to~~ later convenience.)

If we now define

$$U(x) \equiv \exp\left(2i \sum_a \xi_a(x) \lambda_a\right), \text{ then}$$

we can note that $U(x)$ transforms in the $(\bar{3},3)$ representation of $SU(3) \times SU(3)$:

$$U'(x) = \exp\left(i \sum_a \lambda_a \theta_a^R\right) U(x) \exp\left(-i \sum_a \lambda_a \theta_a^L\right)$$

$$\theta_a^R \equiv \theta_a^V - \theta_a^A \quad , \quad \theta_a^L \equiv \theta_a^V + \theta_a^A$$

Note this non-linearly realizes $SU(3) \times SU(3)$ symmetry because components of U satisfy $U^\dagger U = 1$ and $\det U = 1$.

We can now construct the Lagrangian for Goldstone boson interactions. We require the Lagrangian to be invariant under the original global symmetry. Terms with $U^\dagger U$ will be trivial, and $\mathrm{tr}[U^2]$, $\mathrm{tr}[U^4]$ are forbidden, so there is no nontrivial potential energy for U. (Moduli space!)

We can write, at lowest order, $\mathcal{L} = \frac{-1}{16} F^2 \, \mathrm{Tr}\left[\partial_\mu U \partial^\mu U^\dagger\right]$

Following Weinberg,

$$\sum \lambda_a \xi_a = \frac{\sqrt{2}}{F} \begin{bmatrix} \frac{1}{\sqrt{2}}\pi^0 + \frac{1}{\sqrt{6}}\eta^0 & \pi^+ & K^+ \\ \pi^- & \frac{-1}{\sqrt{2}}\pi^0 + \frac{1}{\sqrt{6}}\eta^0 & K^0 \\ \bar{K}^- & \bar{K}^0 & -\sqrt{\frac{2}{3}}\,\eta^0 \end{bmatrix}$$

$$\equiv \frac{\sqrt{2}\, B}{F}$$

Be aware!
Weinberg metric
is $(-+++)$

The prefactors ensure the kinetic terms are canonical:

$$\mathcal{L}_{kin} = -\frac{1}{2} \partial_\mu \pi^0 \partial^\mu \pi^0 - \partial_\mu \pi^+ \partial^\mu \pi^- - \partial_\mu K^+ \partial^\mu \bar{K}^- - \partial_\mu K^0 \partial^\mu \bar{K}^0 - \frac{1}{2} \partial_\mu \eta^0 \partial^\mu \eta^0$$

7/9

This systematizes to higher orders: (originally by Gasser & Leutwyler, 1985)

$$\mathcal{L}_4 = L_1 \,\mathrm{Tr}[\partial_\mu U^\dagger \partial^\mu U]^2 + L_2 \,\mathrm{Tr}[\partial_\mu U \,\partial_\nu U^\dagger]\,\mathrm{Tr}[\partial^\mu U \partial^\nu U^\dagger]$$

$$+ L_3 \,\mathrm{Tr}[\partial_\mu U \partial^\mu U^\dagger \partial_\nu U \partial^\nu U^\dagger]$$

$$+ L_4 \,\mathrm{Tr}[\partial_\mu U \partial^\mu U^\dagger]\,\mathrm{Tr}[M_q(U+U^\dagger)]$$

$$+ L_5 \,\mathrm{Tr}[\partial_\mu U \partial^\mu U^\dagger (M_q U + U^\dagger M_q)]$$

$$+ L_6 \{\mathrm{Tr}[M_q(U+U^\dagger)]\}^2$$

$$+ L_7 \{\mathrm{Tr}[(U^\dagger - U)M_q]\}^2$$

$$+ L_8 \,\mathrm{Tr}[((U M_q)^2 + (U^\dagger M_q)^2)]$$

The first three terms are tested by 4-pion scattering, for example.

Note we have also included the effect of the explicit breaking of the global symmetry by the quark mass matrix:

$$\mathcal{L}_{mass} = -\bar{q} M_q q, \quad M_q = \begin{pmatrix} m_u & 0 & 0 \\ 0 & m_d & 0 \\ 0 & 0 & m_s \end{pmatrix}$$

Following the definition that the Goldstone boson-free quark fields are $\tilde{q}(x)$, we write

$$q(x) \equiv \begin{pmatrix} u(x) \\ d(x) \\ s(x) \end{pmatrix} = \exp\left(-i\gamma^5 \sum_a \xi_a(x) \lambda_a\right) \tilde{q}(x)$$

and

$$\tilde{q}'(x) = \exp\left(i \sum_a \theta_a(x) \lambda_a\right) \tilde{q}(x)$$

We can then observe the consequence of nonzero M_q.

$$\mathcal{L}_{mass} = -\bar{\tilde{q}}\, e^{-i\sqrt{2}\gamma^5 B/F_\pi} M_q\, e^{-i\sqrt{2}\gamma^5 B/F_\pi}\, \tilde{q}$$

We replace the quarks by vevs:

$$\langle \bar{\tilde{q}}_n \gamma^5 \tilde{q}_m \rangle_0 = 0, \qquad \langle \bar{\tilde{q}}_n \tilde{q}_m \rangle_0 = -v\,\delta_{nm}$$

Expanding exponentials to quadratic order in fields, we have an identity that the mass matrix of PNGBs is

$$M^2_{ab} = -\frac{1}{F_\pi^2} \langle \bar{q} \{\lambda_a, \{\lambda_b, M_q\}\} q \rangle_0 .$$

(This follows Sec. 19.3 in Weinberg, Vol. 2, and essentially comes from the symmetric second derivative of the potential expressed as Goldstone perturbations.)

We get

$$\frac{-v}{F_\pi^2} \mathrm{Tr}\{B, \{B, M_q\}\} = \frac{-v}{F_\pi^2} \Big[4m_u \left(\frac{\pi^0}{\sqrt{2}} + \frac{\eta^0}{\sqrt{6}} \right)^2 + 4(m_u+m_d)\pi^+\pi^- + 4(m_u+m_s) K^+ \bar{K}^- + 4m_d \left(\frac{-\pi^0}{\sqrt{2}} + \frac{\eta^0}{\sqrt{6}} \right)^2 + 4(m_d+m_s) K^0 \bar{K}^0 + \frac{8}{3} m_s (\eta^0)^2 \Big]$$

Gives meson masses

$$m^2_{\pi^+} = m^2_{\pi^0} = \frac{4v}{F_\pi^2}(m_u+m_d)$$

$$m^2_{K^+} = \frac{4v}{F_\pi^2}(m_u+m_d)$$

$$m^2_{K^0} = \frac{4v}{F_\pi^2}(m_d+m_s)$$

$$m^2_{\eta^0} = \frac{4v}{F_\pi^2}\left(\frac{m_u+m_d+4m_s}{3}\right)$$

and $m^2_{\pi\eta} = \frac{4v}{\sqrt{3} F_\pi^2}(m_u - m_d)$

The axial current also prescribes an interaction between protons and neutrons.

$$\langle p | A^\mu_+(x) | n\rangle = (2\pi)^{-3} e^{iq\cdot x} \bar{u}_p\left[-i\gamma^\mu\gamma^5 f(q^2) + q^\mu\gamma^5 g(q^2) + iq_\nu[\gamma^\mu,\gamma^5]\gamma_5 \cdot h(q^2)\right]u_n$$

for $q = p_n - p_p$.

For exact $SU(2)\times SU(2)$ for pions, we have $q_\mu\langle p|A^\mu_+(x)|n\rangle = 0$, and from Dirac eqns for u_p and u_n,

$$\bar{u}_p(i\not{p}_p + m_N) = (i\not{p}_n + m_N)u_n = 0,$$

we have $\quad q_\mu \bar{u}_p[-i\gamma^\mu\gamma_5]u_n = -2m_N\bar{u}_p\gamma^5 u_n,$

so $\quad 2m_N f(q^2) = q^2 g(q^2).$

If $g(q^2)$ had no singularity at $q^2=0$, then we must have $m_N = 0$ or $f(0)=0$; we know $m_N \neq 0$ and we conventionally call $f(0) \equiv g_A = 1.2573\,(28)$.

Instead, in limit of exact $SU(2)\times SU(2)$, $g(q^2)$ has a pion pole: $g(q^2) \to \dfrac{2m_N g_A}{q^2}$.

Suppose we write pion-nucleon coupling as $-2iG_{\pi N}\vec{\pi}\,\bar{N}\gamma_5\vec{t}N$, then we would identify the behavior as $q^2\to 0$ from this interaction as

$$G_{\pi N} = \frac{2m_N g_A}{F_\pi},$$ the Goldberger-Treiman relation.

Using $m_N = \left(\dfrac{m_p+m_n}{2}\right) = 938.9\text{ MeV}$, $g_A = 1.257$, $F_\pi = 184\text{ MeV}$, derive $G_{\pi N} \simeq 12.7$, and exp. gives $G_{\pi N} = 13.5$.