QFT3

The Standard Model and EW Theory

Special topic lecture: Chiral anomalies

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After discussing the emergence of flavor-changing neutral currents at 1-loop, we now turn to another topic highly relevant for understanding quantum field theories at 1-loop, namely chiral anomalies.

Recall that chiral symmetries treat LH and RH fermion fields differently. This is clearly evident in EW gauge symmetry, where LH fields interact with $SU(2)$ gauge bosons while RH fields do not, and also with $U(1)_Y$ charges LH and RH fields differently. In contrast, QED and QCD are vector-like symmetries, since LH and RH fields interact the same (same reps. / charges).

In our 1-loop analysis, we were motivated to consider qualitatively new phenomena that was absent at tree-level. With a slight abuse of language, we can ask whether quantum effects introduce qualitatively new phenomena compared to classical expectations. In particular, we can consider the classical versus quantum conservation law for chiral currents.

We recall the Dirac equation.

$$\begin{cases}(i\not\partial - m)\psi = 0\\ -i\,\partial_\mu\bar{\psi}\,\gamma^\mu - m\bar{\psi} = 0\end{cases}$$

Using $\psi = \begin{pmatrix}\psi_L\\ \psi_R\end{pmatrix}$,

$$(i\not\partial - m)\psi = \begin{pmatrix} -m & i(\partial_0+\vec{\sigma}\cdot\vec{\nabla})\\ i(\partial_0-\vec{\sigma}\cdot\vec{\nabla}) & -m\end{pmatrix}\begin{pmatrix}\psi_L\\ \psi_R\end{pmatrix}=0$$

$$\Rightarrow \quad \begin{aligned} i\bar{\sigma}\cdot\partial\psi_L - m\psi_R &= 0\\ i\sigma\cdot\partial\psi_R - m\psi_L &= 0\end{aligned}$$

As usual, if we set $m=0$, then we have separate conservation of LH and RH fermion numbers. Since we write $j^\mu_L = \bar{\psi}\gamma^\mu P_L\psi$ and $j^\mu_R = \bar{\psi}\gamma^\mu P_R\psi$, we have $\partial_\mu j^\mu_L = 0$ and $\partial_\mu j^\mu_R = 0$ for $m=0$. Equivalently, $j^\mu = \bar{\psi}\gamma^\mu\psi$ and $j^{\mu 5} = \bar{\psi}\gamma^\mu\gamma^5\psi$ are both conserved for $m=0$ (at the classical level).

What about at 1-loop order? We can calculate the 1-loop diagrams where $j^{\mu 5}$ is inserted as a vertex coupling via a triangle loop to two massless gauge bosons of a vector symmetry.

1st Method. Follow P&S.

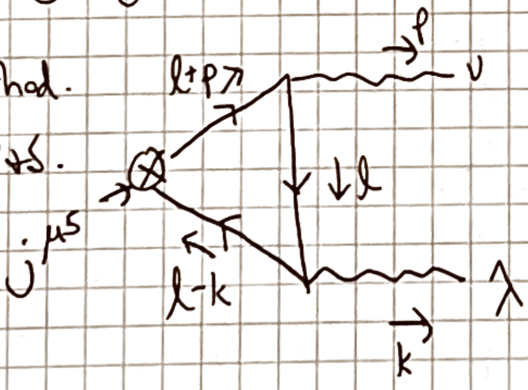


diagram 1

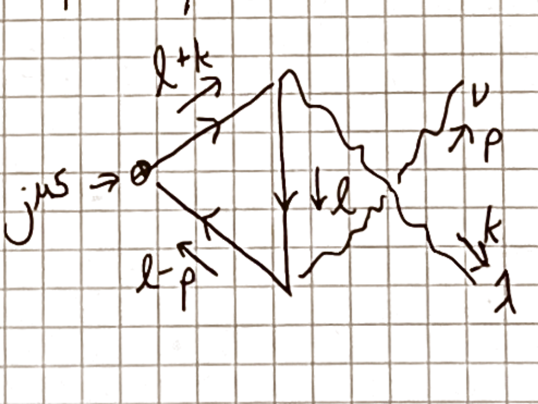


diagram 2

Will assume fermions are massless.

Vtx function,

$$i\mathcal{M}_1^{\mu\nu\lambda} = (-1)\int \frac{d^4 l}{(2\pi)^4}\, \mathrm{Tr}\Big[(i\gamma^\mu\gamma^5)\,\frac{i(\not l - \not k)}{(l-k)^2}\,(-ie\gamma^\lambda) \cdot \frac{i\not l}{l^2}\,(-ie\gamma^\nu)\,\frac{i(\not l+\not p)}{(l+p)^2}\Big]$$

$$i\mathcal{M}_2^{\mu\nu\lambda} = (-1)\int \frac{d^4 l}{(2\pi)^4}\, \mathrm{Tr}\Big[(i\gamma^\mu\gamma^5)\,\frac{i(\not l - \not p)}{(l-p)^2}\,(-ie\gamma^\nu)\, \frac{i\not l}{l^2}\,(-ie\gamma^\lambda)\,\frac{i(\not l+\not k)}{(l+k)^2}\Big]$$

$i\mathcal{M}_2$ is same as $i\mathcal{M}_1$ under simultaneous interchange of $(p,\nu) \leftrightarrow (k,\lambda)$.

To test conservation of the axial vector current, we dot with ~~incomp~~ incoming momentum $q^\mu \equiv p^\mu + k^\mu$.

$$q_\mu\, i\mathcal{M}_1^{\mu\nu\lambda} = (+1)\int \frac{d^4 l}{(2\pi)^4}\, e^2\, \mathrm{Tr}\left[\frac{\not q\,\gamma^5(\not l-\not k)\gamma^\lambda \not l \gamma^\nu(\not l+\not p)}{(l^2)(l-k)^2(l+p)^2}\right]$$

We remark that the integral is linearly divergent using NDA. Even if we drop one power of l in the numerator (since the three powers of l in the numerator being odd), it is log-divergent.

Treating γ^5 naively, we can anticommute it past the loop momenta $\not l$ using the following steps:

Naive! $\not q\,\gamma^5 \overset{\text{safe}}{=} (\not l+\not p-\not l+\not k)\gamma^5 \overset{\text{naive}}{=} (\not l+\not p)\gamma^5 + \gamma^5(\not l-\not k)$

Note $\not a\not a = a_\mu a_\nu \gamma^\mu\gamma^\nu = \frac{1}{2}a_\mu a_\nu(\gamma^\mu\gamma^\nu+\gamma^\nu\gamma^\mu)$
$= \frac{1}{2}a_\mu a_\nu\, 2g^{\mu\nu} = a^2.$

Then, naively, $iq_\mu \mathcal{M}_1^{\mu\nu\lambda} = e^2 \int \frac{d^4 l}{(2\pi)^4} \mathrm{Tr}\left[\frac{(\not{l}+\not{p})\gamma^5(\not{l}-\not{k})\gamma^\lambda \not{l}\gamma^\nu(\not{l}+\not{p})}{l^2(l-k)^2(l+p)^2}\right]$

$$+e^2\int\frac{d^4 l}{(2\pi)^4}\mathrm{Tr}\left[\frac{\gamma^5(\not{l}-\not{k})(\not{l}-\not{k})\gamma^\lambda\not{l}\gamma^\nu(\not{l}+\not{p})}{l^2(l-k)^2(l+p)^2}\right]$$

Use cyclicity

$$=\int\frac{d^4 l}{(2\pi)^4}e^2\left\{\mathrm{Tr}\left[\frac{\gamma^5(\not{l}-\not{k})\gamma^\lambda\not{l}\gamma^\nu}{l^2(l-k)^2}\right]+\mathrm{Tr}\left[\frac{\gamma^5\gamma^\lambda\not{l}\gamma^\nu(\not{l}+\not{p})}{l^2(l+p)^2}\right]\right\}$$

Now, shift first term by $l \to l+k$. Second term: move γ^λ to back of trace.

$$iq_\mu\mathcal{M}_1^{\mu\nu\lambda}=\int\frac{d^4 l}{(2\pi)^4}e^2\left\{\mathrm{Tr}\left[\frac{\gamma^5\not{l}\gamma^\lambda(\not{l}+\not{k})\gamma^\nu}{l^2(l+k)^2}\right]+\mathrm{Tr}\left[-\frac{\gamma^5\not{l}\gamma^\nu(\not{l}+\not{p})\gamma^\lambda}{l^2(l+p)^2}\right]\right\}$$

Now, since second diagram has $(p,\nu)\leftrightarrow(k,\lambda)$, we see that the two diagrams sum to zero.

Now, this naive result is flawed because of the

$\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$ treatment of γ^5 in dim. reg. Recall γ^5 is defined given $\{\gamma^\mu,\gamma^\nu\}=2g^{\mu\nu}$, where μ runs over spacetime indices, and thus the appropriate "γ^5" that differentiates LH and RH spin space (Little group) subgroups would change if we were not in 4D spacetime. To address this issue, a common approach is to adopt the 't Hooft - Veltman scheme for γ^5 and loop momenta. In this ~~scheme~~ scheme, we break up the d-dimensional loop momentum into a 4-dim. piece and a piece that lives in $(d-4)$-dim. Namely,

$$\not{l} = \underset{\substack{\uparrow\\ 4\text{-dim}}}{\not{l}_{\parallel}} + \underset{\substack{\uparrow\\ (d-4)\text{-dim}}}{\not{l}_{\perp}}\ .$$

Then, γ^5 anticommutes with $\not{l}_{\parallel}$ but commutes with $\not{l}_{\perp}$ (more precisely, γ^5 anti-commutes w/ γ-matrices when loop momenta is 4-dim. but commutes with γ-matrices when loop momenta is $(d-4)$-dim.).

So,
$$\not{q}\gamma^5 \equiv (\not{l}+\not{p}-\not{l}+\not{k})\gamma^5$$
$$= (\not{l}_{\parallel}+\not{l}_{\perp}+\not{p}-\not{l}_{\parallel}-\not{l}_{\perp}+\not{k})\gamma^5$$
$$= (\not{l}+\not{p})\gamma^5+\gamma^5\not{l}_{\parallel}-\gamma^5\not{l}_{\perp}-\gamma^5\not{k}$$
$$= (\not{l}+\not{p})\gamma^5+\gamma^5\not{l}_{\parallel}+\gamma^5\not{l}_{\perp}-\gamma^5\not{k}-2\gamma^5\not{l}_{\perp}$$
$$= (\not{l}+\not{p})\gamma^5+\gamma^5(\not{l}-\not{k})-2\gamma^5\not{l}_{\perp}$$

Therefore, we see that, compared to the naive result, we have an additional term in $iq_\mu \mathcal{M}_1^{\mu\nu\lambda}$:
$$iq_\mu \mathcal{M}_1^{\mu\nu\lambda} = (+1)e^2\int\frac{d^dl}{(2\pi)^d}\,\mathrm{Tr}\left[\frac{-2\gamma^5\not{l}_{\perp}(\not{l}-\not{k})\gamma^\lambda\not{l}\gamma^\nu(\not{l}+\not{p})}{l^2(l-k)^2(l+p)^2}\right]$$

Solve explicitly using usual techniques of dim. reg.

1. Combine denominators using Feynman params.

Use $k^2=p^2=0$.
$$\frac{1}{D_1D_2D_3} = 2\int dx\,dy\,dz\,\frac{\delta(x+y+z-1)}{(zl^2+x(l-k)^2+y(l+p)^2)^3}$$
$$= 2\int dx\,dy\,\frac{1}{(l^2-2xl\cdot k+2yl\cdot p)^3}$$

2. Shift $L\equiv l-xk+yp$.

Then $l^2-2xl\cdot k+2yl\cdot p = L^2-\Delta$,

for $\Delta\equiv(xk-yp)^2 = -2xyk\cdot p$.
$$iq_\mu \mathcal{M}_1^{\mu\nu\lambda} = e^2\int\frac{d^dl}{(2\pi)^d}\int dx\,dy\,\frac{\mathrm{Tr}[-4\gamma^5\not{l}_{\perp}(\not{l}-\not{k})\gamma^\lambda\not{l}\gamma^\nu(\not{l}+\not{p})]}{(L^2-\Delta)^3}$$

Let $P \equiv -xk + yp$, then $L = l + P$, $L - P \equiv l$,

$$\mathrm{Tr}\left[-4\gamma^5 \not{l}_\perp (\not{L} - \not{P} - \not{k})\gamma^\lambda (\not{L} - \not{P})\gamma^\nu (\not{L} - \not{P} + \not{p})\right]$$

Leading power in L:

$$\mathrm{Tr}\left[-4\gamma^5 \not{l}_\perp \not{L}\gamma^\lambda \not{L}\gamma^\nu \not{L}\right]$$

Use $\not{L}\gamma^\nu = 2L^\nu - \gamma^\nu \not{L}$

$$= \mathrm{Tr}\left[-4\gamma^5 \not{l}_\perp \not{L}\gamma^\lambda (2L^\nu)\not{L}\right] - \mathrm{Tr}\left[-4\gamma^5 \not{l}_\perp \not{L}\gamma^\lambda\gamma^\nu\right]L^2$$

First term vanishes, since $\epsilon^{\mu\cdots}$ contracts symmetric $\not{L}\not{L}$ γ-matrices.

Second term vanishes, since evenness of integral requires $\mathrm{Tr}\left[-4\gamma^5 \not{l}_\perp \not{l}_\perp \gamma^\lambda\gamma^\nu\right]L^2$ but then $\mathrm{Tr}\left[\gamma^5\gamma^\lambda\gamma^\nu\right] = 0$.

Two powers in L:

$$\mathrm{Tr}\left[-4\gamma^5 \not{l}_\perp \not{L}\gamma^\lambda (-\not{P})\gamma^\nu (-\not{P} + \not{p})\right]$$
$$+ \mathrm{Tr}\left[-4\gamma^5 \not{l}_\perp (-\not{P} - \not{k})\gamma^\lambda \not{L}\gamma^\nu (-\not{P} + \not{p})\right]$$
$$+ \mathrm{Tr}\left[-4\gamma^5 \not{l}_\perp (-\not{P} - \not{k})\gamma^\lambda (-\not{P})\gamma^\nu \not{L}\right]$$

All $\not{L} \to \not{l}_\perp$ to get even integrand under $d^d l$ integration.

Also, $\not{l}_\perp$ anticommutes w/ other Dirac matrices $\{\not{l}_\perp, \gamma^\mu\} = 0$ as long as other Dirac matrices live in 4D.

Combine $\not{l}_\perp$ in trace, then know result is antisymmetric in γ-matrices.

$$= \mathrm{Tr}\left[-4\gamma^5 \, l_\perp^2 \, \gamma^\lambda (-\not{p}) \gamma^\nu \not{p} \right]$$
$$+ \mathrm{Tr}\left[-4\gamma^5 \, l_\perp^2 \, (-\not{p} - \not{k}) \gamma^\lambda \gamma^\nu (-\not{p} + \not{p}) \right]$$
$$+ \mathrm{Tr}\left[-4\gamma^5 \, l_\perp^2 \, (-\not{k}) \gamma^\lambda (-\not{p}) \gamma^\nu \right]$$

Can show $\not{p}$-dependent terms in second trace vanish, against $\not{p}$-dependent terms in first + third traces.

$$= \mathrm{Tr}\left[-4\gamma^5 \, l_\perp^2 \, (-\not{k}) \gamma^\lambda \gamma^\nu \not{p} \right]$$
$$= -16 i \, l_\perp^2 \, \epsilon^{\alpha\lambda\nu\beta} k_\alpha p_\beta$$

Finally, if no powers of L in trace, then oddness of int. over $l_\perp$ vanishes.

$$i q_\mu M_1^{\mu\nu\lambda} = \frac{e^2}{(2\pi)^d} \int \frac{d^d l \; dx\,dy}{(L^2-\Delta)^3} \, (-16i) \, l_\perp^2 \, \epsilon^{\alpha\lambda\nu\beta} k_\alpha p_\beta$$

Use $l_\perp^2 = \frac{(d-4)}{d} L^2$ and $\int \frac{d^d L}{(2\pi)^d} \frac{L^2}{(L^2-\Delta)^3}$

$$= \frac{(-1)^2}{(4\pi)^{d/2}} \frac{i d}{2} \frac{\Gamma(2-d/2)}{\Gamma(3)} \left(\frac{1}{\Delta}\right)^{2-d/2}$$

$$i q_\mu M_1^{\mu\nu\lambda} = e^2 \int dx\,dy \, \frac{(-16i)}{16\pi^2} \epsilon^{\alpha\lambda\nu\beta} k_\alpha p_\beta \cdot \left(\frac{d-4}{d}\right) \frac{i d}{2} \frac{1}{2} \left(\frac{2}{\epsilon} - \log\Delta - \gamma + \log 4\pi \right)$$

for $d = 4 - \epsilon$.

Use $\int_0^1 dx \int_0^{1-x} dy = \frac{1}{2}$

$$= e^2 \int dx\,dy \left(\frac{+1}{\pi^2} \right) \frac{1}{4} \cdot \epsilon^{\alpha\lambda\nu\beta} k_\alpha p_\beta \, (-\epsilon) \left(\frac{2}{\epsilon} + \ldots \right)$$

$$= \frac{-e^2}{4\pi^2} \epsilon^{\alpha\lambda\nu\beta} k_\alpha p_\beta$$

We get pure const since $\frac{d-4}{\epsilon}$ is exactly $\frac{\epsilon}{\epsilon}$.

$$i q_\mu M_1^{\mu\nu\lambda} = \frac{e^2}{4\pi^2} \epsilon^{\alpha\lambda\beta\nu} k_\alpha p_\beta .$$

Since second term is same under $(p,\nu) \leftrightarrow (k,\lambda)$ interchange,

$$i q_\mu M_{tot}^{\mu\nu\lambda} = \frac{e^2}{2\pi^2} \epsilon^{\alpha\lambda\beta\nu} k_\alpha p_\beta . = \langle \partial_\mu j^{\mu 5} \rangle$$

Importantly, we have calculated a nonzero expectation value for the conservation law of the axial current! Recognizing that $\epsilon^{\alpha\lambda\beta\nu} k_\alpha p_\beta$ is the Feynman rule for $F_{\mu\nu}\tilde{F}^{\mu\nu}$, we write

$$\langle \partial_\mu j^{\mu 5} \rangle = \langle \frac{-e^2}{16\pi^2} \epsilon^{\alpha\nu\beta\lambda} F_{\alpha\nu} F_{\beta\lambda} \rangle .$$

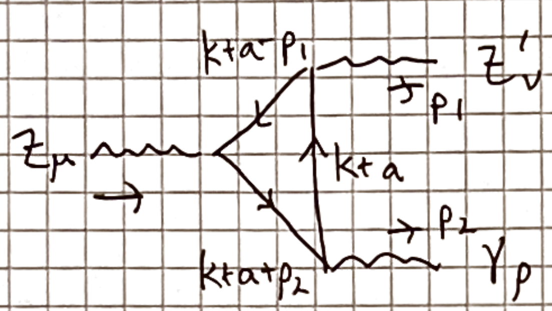
It turns out this is true as an operator eqn., using Fujikawa's method of the transform of the fermion measure.

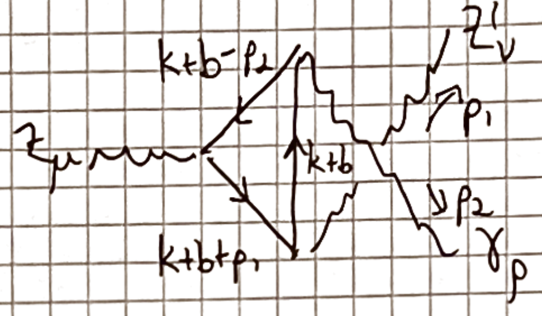
While this is the standard textbook approach to calculating the Adler-Bell-Jackiw chiral anomaly, it is not precise enough to be useful in a modern context. Specifically:

- What if fermions are not massless?
- What if multiple gauge bosons have γ^5 couplings? If multiple gauge bosons have masses?
- What physical observable is measured by a non-vanishing chiral anomaly?
- What if we do not use 't-Hooft - Veltman for γ^5?

Hence, a modern discussion of the chiral anomaly and its phenomenology is in Michaels, Yu, JHEP 03 (2021) 120; [2010.00021]. Based on Dedes, Suxho, PRD 85 (2012) 095024, [1202.4940]

We revisit the triangle diagrams.





Following Weinberg (Vol. II), we allow each diagram to shift the loop momentum independently in dim. reg., so the first diagram uses $k+a$ and the second uses $k+b$.

We simplify the conservation tests of the currents by setting $b=-a$. ~~For~~ Furthermore, we can expand $a_\mu = z\, p_{1\mu} + w\, p_{2\mu}$.

Following the Dedes & Suxho (and Adler method by redefining the vtx. fcn.), we can calculate the 4-divergences of the external currents.

$$(p_1+p_2)_\mu \Gamma^{\mu\nu\rho} = \frac{Qegg_X}{4\pi^2 c_W}\, \epsilon^{\nu\rho|p_1||p_2|}\Big((w-z)(g_V^{Z'}g_A^{Z}+g_V^{Z}g_A^{Z'}) + 4m^2 g_V^{Z'}g_A^{Z}C_0(m)\Big)$$

$$-p_{1\nu}\Gamma^{\mu\nu\rho} = \frac{Qegg_X}{4\pi^2 c_W}\, \epsilon^{\mu\rho|p_1||p_2|}\Big((w-1)(g_V^{Z'}g_A^{Z}+g_V^{Z}g_A^{Z'}) - 4m^2 g_V^{Z}g_A^{Z'}C_0(m)\Big)$$

$$-p_{2\rho}\Gamma^{\mu\nu\rho} = \frac{Qegg_X}{4\pi^2 c_W}\, \epsilon^{\mu\nu|p_1||p_2|}\Big((z+1)(g_V^{Z'}g_A^{Z}+g_V^{Z}g_A^{Z'})\Big)$$

If we normalize all couplings, and set the fermion mass to zero, we get

$$(p_1+p_2)_\mu \Gamma^{\mu\nu\rho} = \frac{\epsilon^{\nu\rho|p_1||p_2|}}{4\pi^2}(w-z)\left(g_V^{z'}g_A^{z}+g_V^{z}g_A^{z'}\right)$$

$$-p_{1\nu}\Gamma^{\mu\nu\rho} = \frac{\epsilon^{\mu\rho|p_1||p_2|}}{4\pi^2}(w-1)\left(g_V^{z'}g_A^{z}+g_V^{z}g_A^{z'}\right)$$

$$-p_{2\rho}\Gamma^{\mu\nu\rho} = \frac{\epsilon^{\mu\nu|p_1||p_2|}}{4\pi^2}(z+1)\left(g_V^{z'}g_A^{z}+g_V^{z}g_A^{z'}\right)$$

In the standard calculation, $w=-z=1$ so the only nonzero divergence is in the μ-vtx. But w + z are arbitrary parameters + should not have physical consequences! Therefore, instead of choosing specific w+z, we want to construct observables (like exotic decay widths of Z or Z') that sum over sets of fermions s.t. the dependence on w + z in the observable vanishes.

The ABJ anomaly encodes the chiral anomaly purely in the μ-vertex (and takes $g_A^{z'} \to 0$ in the above calculation).

How does the calculation change when we use different gauge bosons on the external legs? In analogy to the vacuum polarization diagram for photons vs. gluons, the group space in the trace over virtual fermions factorizes from the momentum space integral.

We can see that this leads to a multiplication of the first diagram by

$Tr[\, t^a t^b t^c]$ and the second diagram by $Tr[\, t^a t^c t^b]$, so overall, we can define the <u>anomaly coefficient</u> $A^{abc} = Tr[\, t^a \{t^b, t^c\}]$.

Therefore, to have consistent chiral current conservation at the quantum level, we need to ensure that sums of chiral matter fermions have vanishing anomaly coefficients.

Let us evaluate the anomaly coefficients for the SM. We will write the fermions in their standard form and also as LH ~~fermions~~ representations.

	SU(3)	SU(2)	U(1)
Q_L	3	2	1/6
u_R	3	1	2/3
d_R	3	1	-1/3
L_L	1	2	-1/2
e_R	1	1	-1
$(u_R)^c$	$\bar{3}$	1	-2/3
$(d_R)^c$	$\bar{3}$	1	+1/3
$(e_R)^c$	1	1	-1

We write all fermions as LH fields because it ensures there are no relative signs between reps. + conj. reps.

We have a long list of possible gauge anomaly coefficients to check. We also have to include gravitational anomalies, which is $\Sigma\, U(1)_Y$. (See P+S, Chap. 19.)

These are

$SU(3)_c^3$ $\qquad$ $SU(3)_c \times SU(2)^2$

$SU(3)_c^2 \times SU(2)$ $\qquad$ $SU(3)_c \times SU(2) \times U(1)_Y$

$SU(3)_c^2 \times U(1)_Y$ $\qquad$ $SU(3)_c \times U(1)_Y^2$

$SU(2)^3$ $\qquad$ $SU(2) \times U(1)_Y^2$ $\qquad$ $\Sigma\, U(1)_Y$

$SU(2)^2 \times U(1)_Y$ $\qquad$ $U(1)_Y^3$ $\qquad$ $= U(1) \times \text{grav}^2$

Now, since A^{abc} is a trace over group generators, we can immediately note that the trace vanishes if we include only one $SU(N)$ group, since all $SU(N)$ generators are traceless. Also, $SU(3)^3$ anomaly coefficient is fermion by fermion zero, since

For Fundamental reps. of $SU(N)$

$$A^{abc} = \mathrm{Tr}[\, t^a \{t^b, t^c\}] = \mathrm{Tr}\left[\, t^a \left(\tfrac{1}{N}\delta^{bc} + d^{bcx}\, t_N^x\right)\right] \qquad \text{P+S, 19.138}$$

$$= \mathrm{Tr}\left[\, t^a t_N^x\, d^{bcx}\right]$$

and fermions are vectorlike under $SU(3)_c$.

$SU(2)^3$ vanishes because

$$\mathrm{Tr}\left[\tau^a\{\tau^b, \tau^c\}\right] = \frac{1}{8}\,\mathrm{tr}\left[\sigma^a\{\sigma^b, \sigma^c\}\right]$$

$$= \frac{1}{8}\,\mathrm{tr}\left[\sigma^a\, 2\delta^{bc}\right] = 0.$$

because $d^{abc} = 0$ for $SU(2)$.

Remains to check:

$SU(3)^2 \times U(1)_Y$

$$\underset{\substack{\uparrow \\ \#\ \text{generations}}}{3} \cdot \underset{\substack{\uparrow \\ SU(2)\ \text{multiplicity}}}{2} \cdot \underbrace{\mathrm{tr}\left[\mathbb{1}\{t^b, t^c\}\right]\frac{1}{6}}_{Q_L} + \underbrace{3\,\mathrm{tr}[\ldots]\left(-\frac{2}{3}\right)}_{u_R^c} + \underbrace{3\,\mathrm{tr}[\ldots]\left(+\frac{1}{3}\right)}_{d_R^c} = 0$$

$SU(2)^2 \times U(1)_Y$:

$$3 \cdot \underset{\substack{\uparrow \\ \#\ \text{colors}}}{3} \cdot \underbrace{\mathrm{tr}\left[\mathbb{1}\{\tau^b, \tau^c\}\right] \cdot \frac{1}{6}}_{Q_L} + \underbrace{3\,\mathrm{tr}\left[\mathbb{1}\{\tau^b, \tau^c\}\right]\left(-\frac{1}{2}\right)}_{L_L} = 0$$

$U(1)^3$: Related to Diophantine eqns

$$3 \cdot 3 \cdot 2 \cdot \left(\frac{1}{6}\right)^3 + 3 \cdot 3\left(-\frac{2}{3}\right)^3 + 3 \cdot 3\left(\frac{1}{3}\right)^3 + 3 \cdot 2\left(-\frac{1}{2}\right)^3 + 3 \cdot 1 \cdot (1)^3$$

$$= 3\left(\frac{1}{36} - \frac{8}{9} + \frac{1}{9} - \frac{1}{4} + 1\right) = 0$$

$\Sigma\, U(1)$:

$$3 \cdot 3 \cdot 2 \cdot \left(\frac{1}{6}\right) + 3 \cdot 3\left(-\frac{2}{3}\right) + 3 \cdot 3 \cdot \frac{1}{3} + 3 \cdot 2\left(-\frac{1}{2}\right) + 3 \cdot 1\,(1)$$

$$= 3\,(1 - 2 + 1 - 1 + 1) = 0$$

Note: anomalies req. # quarks = # leptons, but not # generations in SM.

Anomaly cancellation is guaranteed if $\exists$ GUT embedding.

Global symmetries can be anomalous.

$$A\left(SU(2)^2 \times U(1)_B\right) = +A\left(SU(2)^2 \times U(1)_L\right) = \frac{3}{2}$$

$$A\left(U(1)_Y^2 \times U(1)_B\right) = A\left(U(1)_Y^2 \times U(1)_L\right) = -3/2$$

So $U(1)_{B-L}$ is anomaly-free. Need RH ν's for $U(1)_L^3$ cancellation.