Lecture 13.

QFT III: The Standard Model and EW Theory

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January 13, 2021

Today: Continue collinear singularities in initial state
+ connect to photon / gluon PDFs.

Last time, we saw the Weizsäcker-Williams equivalent photon approximation

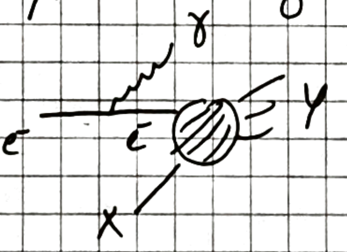
$$\sigma(e^- X \to e^- Y) = \int_0^1 dz \, \frac{\alpha}{2\pi} \log \frac{s}{m^2} \left(\frac{1+(1-z)^2}{z} \right) \sigma(\gamma X \to Y)$$

$$f_\gamma(z) = \frac{\alpha}{2\pi} \log \frac{s}{m^2} \left(\frac{1+(1-z)^2}{z} \right)$$

carries momentum zp

z is the fractional momentum.

Analogously, the diagram with



$$\sigma = \frac{1}{(1+v_X) 2p \, 2E_X} \int \frac{d^3 q}{(2\pi)^3} \frac{1}{2q^0} \int d\Pi_Y \left[\frac{1}{2} \sum |\mathcal{M}|^2 \right] \left(\frac{1}{k^2} \right)^2 |\mathcal{M}_{e^- X}|^2$$

We do the same analysis to account for the virtual e^- + real photon emission:

$$\sigma(e^- X \to \gamma Y) = \int \frac{dz \, dp_\perp^2}{16\pi^2 z} \left[\frac{1}{2} \sum |\mathcal{M}|^2 \right] \frac{z^2}{p_\perp^4} (1-z) \, \sigma(e^- X \to Y)$$

$$= \int \frac{dz \, dp_\perp^2}{16\pi^2} \frac{z(1-z)}{p_\perp^4} \frac{2e^2 p_\perp^2}{z(1-z)} \left[\frac{1+(1-z)^2}{z} \right] \sigma(e^- X \to Y)$$

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$$= \int_0^1 dz \int \frac{dp_\perp^2}{p_\perp^2} \frac{\alpha}{2\pi} \left(\frac{1+(1-z)^2}{z} \right) \sigma(e^- X \to Y)$$

$\uparrow$ carries longitudinal fraction $(1-z)$.

Naively, $\frac{\alpha}{2\pi} \log \frac{s}{m^2} \left(\frac{1+x^2}{1-x} \right) \doteq f_e^{(1)}(x)$

with $x = (1-z)$ is the PDF for electron parton in the electron. But more correct to treat $f_e^{(1)}(x)$ as the $\mathcal{O}(\alpha)$ correction to the "bare" process with no radiation, $f_e^{(0)}(x) = \delta(1-x)$.

Need to fix two issues: divergence at $x=1$ + movement to x from $x=1$ by radiation

Divergence at $x=1$ corresponds to emission of soft photons.

Ansatz: To $\mathcal{O}(\alpha)$,

$$f_e(x) = \delta(1-x) + \frac{\alpha}{2\pi} \log\left(\frac{s}{m^2}\right) \left(\frac{1+x^2}{1-x} - A\,\delta(1-x) \right)$$

How to fix A?

If we ignore pair-creation, $\int_0^1 dx\, f_e(x) = 1$ to conserve electron probability.

How to handle singular denominator?

Use subtraction procedure:

$$\frac{1}{(1-x)_+} = \frac{1}{1-x} \quad \text{for } x<1$$

+ for $x=1$, ~~cancels st~~ matches singularity behavior $\int_0^1 dx \frac{f(x)}{1-x_+} = \int_0^1 dx \frac{f(x)-f(1)}{1-x}$

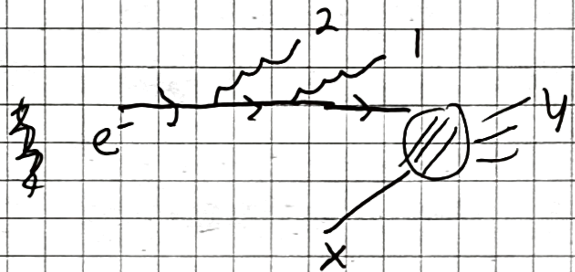
Less formally, P+S (17.106)

$$\frac{1}{1-x_+} = \lim_{\epsilon \to 0} \left[\frac{1}{1-x} \theta(1-x-\epsilon) - \delta(1-x) \int_0^{1-\epsilon} dx' \frac{1}{1-x'} \right]$$

This moves the singular $\frac{1}{1-x}$ to $\frac{1}{1-x_+}$. For normalization, $\int_0^1 dx \, \frac{1+x^2}{(1-x)_+} = \int dx \, \frac{x^2-1}{1-x} = -\frac{3}{2}$

$$\Rightarrow \quad f_e(x) = \delta(1-x) + \frac{\alpha}{2\pi}\log\frac{s}{m^2}\left(\frac{1+x^2}{(1-x)_+} + \frac{3}{2}\delta(1-x)\right)$$

Properly normalized, but highly singular at $x=1$.
Singularity is attributed to many collinear photons.



Analyze $p_{\perp,2}$: for $p_{\perp,2} \ll p_{\perp,1}$, the virtuality of the e^- legs can be matched. We get

$$\left(\frac{\alpha}{2\pi}\right)^2 \int_{m^2}^{s} \frac{dp_{\perp,1}^2}{p_{\perp,1}^2} \int_{m^2}^{p_{\perp,1}^2} \frac{dp_{\perp,2}^2}{p_{\perp,2}^2} = \frac{1}{2}\left(\frac{\alpha}{2\pi}\right)^2 \log^2\left(\frac{s}{m^2}\right)$$

For $p_{\perp,2} \gg p_{\perp,1}$, there is no double log. Only for $p_{\perp,2} \ll p_{\perp,1}$ does the collinear splitting at $\mathcal{O}(\alpha^2)$ compete with $\mathcal{O}(\alpha)$.

Iterate: for n splittings, we get

$$\frac{1}{(n!)}\left(\frac{\alpha}{2\pi}\right)^n \log^n\left(\frac{s}{m^2}\right).$$

This enhancement should be resummed.

Since each $p_{\perp,k} \gg p_{\perp,k+1}$, the virtual momenta are increasingly off-shell as we proceed from the external leg toward the hard collision. This is called "strongly ordered."

Intuitively, we are probing the intermediate electron at progressively higher virtuality.

Consequently, we can focus on the immediate description that relates a given electron parton + photon parton to a related set of partons at a small difference in ΔQ. We can take $f_\gamma(x, Q)$ and $f_e(x, Q)$ for the probabilities of having a γ or e^- of long. fraction x, accounting for collinear photons with $p_\perp < Q$. Then, if we raise Q to $Q+\Delta Q$, then $f_e(x, Q)$ can source a photon with $Q < p_\perp < Q+\Delta Q$. Using the splitting probability of $\frac{\alpha}{2\pi}\frac{dp_\perp^2}{p_\perp^2}\frac{1+(1-z)^2}{z}$,

we have the ~~correction~~ computation to $f_\gamma(x, Q+\Delta Q)$ as

$$
\begin{aligned}
f_\gamma(x, Q+\Delta Q) &= f_\gamma(x, Q) + \int_0^1 dx' \int_0^1 dz \left[\frac{\alpha}{2\pi}\frac{(\Delta Q)^2}{Q^2}\frac{1+(1-z)^2}{z}\right] f_e(x', p_\perp)\,\delta(x-zx') \\
&= f_\gamma(x, Q) + \frac{\Delta Q}{Q}\int_z^1 \frac{dz}{z}\left[\frac{\alpha}{\pi}\frac{1+(1-z)^2}{z}\right] f_e\left(\frac{x'}{z}, p_\perp\right)
\end{aligned}
$$

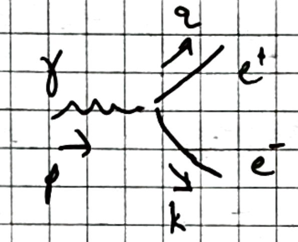
We can compactly express this result using an integral-differential equation, where the splitting function serves as the kernel for the evolution

$$
\frac{d}{d\log Q} f_\gamma(x, Q) = \int_z^1 \frac{dz}{z}\left[\frac{\alpha}{\pi}\frac{1+(1-z)^2}{z}\right] f_e\left(\frac{x}{z}, Q\right)
$$

This allows us to evolve up & down in Q. The corresponding fcn. for electrons is

$$
\frac{d}{d\log Q} f_e(x, Q) = \int_x^1 \frac{dz}{z}\left[\frac{\alpha}{\pi}\left(\frac{1+z^2}{(1-z)_+} + \frac{3}{2}\delta(1-z)\right)\right] f_e\left(\frac{x}{z}, Q\right)
$$

To complete our analysis at fixed order in α, we also have to include photon splitting into e^+e^- pairs. By analyzing the diagram



, and separating into helicities,

we get $\frac{1}{2}\sum_{pols} |\mathcal{M}|^2 = \frac{2e^2 p_\perp^2}{z(1-z)}\left[z^2+(1-z)^2\right]$

in the collinear region. Including this in the evolution equations for QED gives the following, first given by Gribov and Lipatov:

$$\frac{d}{d\log Q} f_\gamma(x,Q) = \frac{\alpha}{\pi}\int_x^1 \frac{dz}{z}\left[P_{\gamma\leftarrow e}(z)\left(f_e+f_{\bar{e}}\right)+P_{\gamma\leftarrow\gamma} f_\gamma\right]$$

$$\frac{d}{d\log Q} f_e = \frac{\alpha}{\pi}\int_x^1 \frac{dz}{z}\left[P_{e\leftarrow e}(z) f_e\left(\tfrac{x}{z},Q\right)+P_{e\leftarrow\gamma} f_\gamma\left(\tfrac{x}{z},Q\right)\right]$$

$$\frac{d}{d\log Q} f_{\bar{e}} = \frac{\alpha}{\pi}\int_x^1 \frac{dz}{z}\left[P_{e\leftarrow e}(z) f_{\bar{e}}\left(\tfrac{x}{z},Q\right)+P_{e\leftarrow\gamma} f_\gamma\left(\tfrac{x}{z},Q\right)\right]$$

where the splitting kernels give the probability of getting parton x from y in $P_{x\leftarrow y}$,

$$P_{e\leftarrow e}(z) = \frac{1+z^2}{(1-z)_+} + \frac{3}{2}\delta(1-z)$$

$$P_{\gamma\leftarrow e}(z) = \frac{1+(1-z)^2}{z}$$

$$P_{e\leftarrow\gamma}(z) = z^2+(1-z)^2$$

$$P_{\gamma\leftarrow\gamma}(z) = -\frac{2}{3}\delta(1-z)$$

When we pass to QCD, we have the same analysis with the complications of group factors & the gluon self-interaction. These are the Altarelli-Parisi evolution equations:

$$\frac{d}{d\log Q} f_g(x,Q) = \frac{\alpha_s(Q^2)}{\pi}\int_x^1 \frac{dz}{z}\Big[P_{g\leftarrow q}(z)\sum_f [f_f(\tfrac{x}{z},Q) + f_{\bar f}(\tfrac{x}{z},Q)] + P_{g\leftarrow g}(z)\, f_g(\tfrac{x}{z},Q)\Big]$$

$$\frac{d}{d\log Q} f_f(x,Q) = \frac{\alpha_s(Q^2)}{\pi}\int_x^1 \frac{dz}{z}\Big[P_{q\leftarrow q}(z)\, f_f(\tfrac{x}{z},Q) + P_{q\leftarrow g}(z) f_g\Big]$$

$$\frac{d}{d\log Q} f_{\bar f}(x,Q) = \frac{\alpha_s(Q^2)}{\pi}\int_x^1 \frac{dz}{z}\Big[P_{q\leftarrow q}(z) f_{\bar f}(\tfrac{x}{z},Q) + P_{q\leftarrow g}(z) f_g\Big]$$

with the splitting kernels:

$$P_{q\leftarrow q}(z) = \frac{4}{3}\Big[\frac{1+z^2}{(1-z)_+} + \frac{3}{2}\delta(1-z)\Big]$$

$$P_{g\leftarrow q}(z) = \frac{4}{3}\Big[\frac{1+(1-z)^2}{z}\Big]$$

$$P_{q\leftarrow g}(z) = \frac{1}{2}\Big[z^2+(1-z)^2\Big]$$

$$P_{g\leftarrow g}(z) = 6\Big[\frac{1-z}{z} + \frac{z}{(1-z)_+} + z(1-z) + \Big(\frac{11}{12} - \frac{n_f}{18}\Big)\delta(1-z)\Big]$$

Knowing a set of extracted PDFs for quarks & gluon at scale Q, we can evolve & predict PDFs at a new scale using the Altarelli-Parisi evolution equations to $\mathcal{O}(\alpha_s)$ accuracy.