Lecture 12.
QFT 3: The Standard Model & EW Theory.
January 7, 2021
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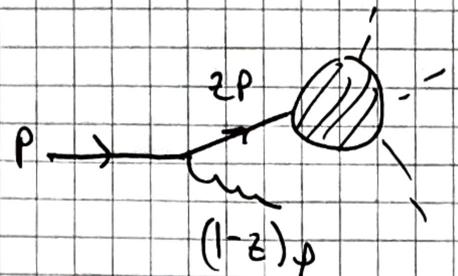
Last time: Kinematics & dijet production
Today: KLN theorem
PDFs & collinear divergences
Emergence of photon PDF from electron
DGLAP equations

Recall Kinoshita-Lee-Nauenberg theorem:
SM is IR-finite. IR divergences from loop integrals are cancelled by IR divergences in phase space integrals.

Technical discussion of $\frac{1}{\epsilon^2}$ vs. $\frac{1}{\epsilon}$ IR poles is given in Schwartz, Sec. 32.2. Here, I will summarize / give an overview of the structure then follow P+S for motivating & deriving DGLAP. Essentially, all soft & collinear IR divergences are canceled by including real emission & virtual diagrams, except for collinear singularities in initial ~~& final states~~ states. (Should also include external leg corrections.)

Why no soft divergence in ISR splitting? From G. Salam, Lectures on QCD #2, PDF @ Bautzen in 2009, 2

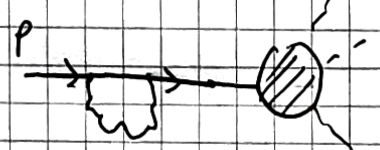


Hard process after splitting, $p \to zp$.

$$\sigma_g(p) \triangleq \sigma_h(zp)\,\frac{\alpha_s C_F}{\pi}\,\frac{dz}{1-z}\,\frac{dk_t^2}{k_t^2}$$

For virtual terms, momentum entering hard process is unchanged.

$$\sigma_v(p) \triangleq -\sigma_h(p)\,\frac{\alpha_s C_F}{\pi}\,\frac{dz}{1-z}\,\frac{dk_t^2}{k_t^2}$$



Total xsec gets contribution with two different hard processes:

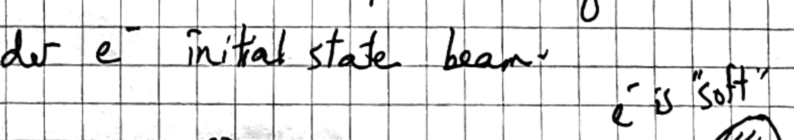
$$\sigma_g + \sigma_v \triangleq \frac{\alpha_s}{\pi} C_F \int \frac{dk_t^2}{k_t^2}\,\frac{dz}{1-z}\left(\sigma_h(zp) - \sigma_h(p)\right)$$

In soft limit, $z \to 1$ + soft divergence cancels.
For $z \neq 1$, the diff. $\sigma_h(zp) - \sigma_h(p)$ is finite + non-vanishing. We still have a collinear divergence as $k_t \to 0$.

So, we focus only on the collinear divergence in ISR.

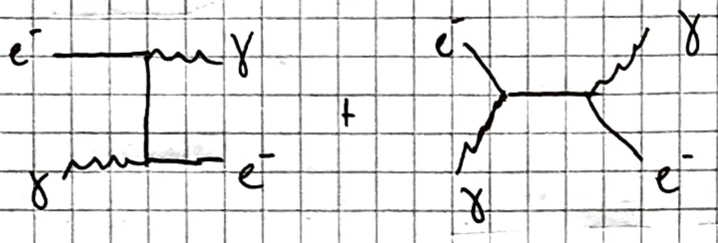
Start with QED + ep scattering, from P+S, 17.5.
Consider e^- initial state beam:

e^- is "soft" ($z \ll 1$)

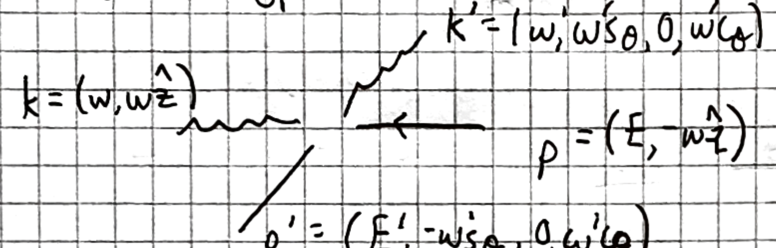


IR divergent collinear photon ⇒ can be "hard"!

Consider Compton scattering.



High energy behavior: in COM frame.



$$p \cdot k' = E\omega + \omega^2 c_\theta = \omega(E + \omega c_\theta)$$

$$p \cdot k = E\omega + \omega^2$$

Show $\omega' = \omega$, $E' = E$

$$\Rightarrow \omega + E = \omega' + E'$$

$$\omega + \sqrt{\omega^2 + m^2} = \omega' + \sqrt{\omega'^2 + m^2}$$

$$\omega^2 + 2\omega\sqrt{\omega^2+m^2} + \omega^2 + m^2 = \omega'^2 + 2\omega'\sqrt{\omega'^2+m^2} + \omega'^2 + m^2$$

$$2\omega\left(\omega + \sqrt{\omega^2+m^2}\right) = 2\omega'\left(\omega' + \sqrt{\omega'^2+m^2}\right)$$

Use first eqn $\Rightarrow$ $\omega = \omega'$.

We have the $\frac{1}{4}\sum_s |\mathcal{M}|^2 = 2e^4\left[\frac{p\cdot k'}{p\cdot k} + \frac{p\cdot k}{p\cdot k'} + 2m^2\left(\frac{1}{p\cdot k} - \frac{1}{p\cdot k'}\right) + m^4\left(\frac{1}{p\cdot k} - \frac{1}{p\cdot k'}\right)^2\right]$

Drop m^2 & m^4 in numerator since we want to analyze singularities at $\theta \approx \pi$.

$$\frac{1}{4}\sum_s |\mathcal{M}|^2 = 2e^4\left[\frac{\omega(E+\omega c_\theta)}{\omega(E+\omega)} + \frac{\omega(E+\omega)}{\omega(E+\omega c_\theta)}\right]$$

first term is regular as $c_\theta \to -1$, discard.

$$= \frac{2e^4(E+\omega)}{E+\omega c_\theta}$$

Then $$\frac{d\sigma}{dc_\theta} = \frac{1}{2}\cdot\frac{1}{2E}\cdot\frac{1}{2\omega}\cdot\frac{\omega}{(2\pi)\,4(E+\omega)}\;\frac{2e^4(E+\omega)}{(E+\omega c_\theta)}$$

$$\approx \frac{2\pi\alpha^2}{2m^2 + s(1+c_\theta)}$$

For High energy scattering, $s \gg m^2$,
denominator vanishes for $\theta \approx \pi$.

Impose cutoff by m_e (since $E = \sqrt{\omega^2 + m_e^2}$)

$$\int_{-1}^{1} dc_\theta \, \frac{d\sigma}{dc_\theta} \approx \frac{2\pi\alpha^2}{s} \int_{-1}^{1} dc_\theta \, \frac{1}{1+c_\theta}$$

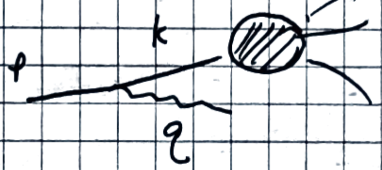
$$\sigma_{tot} \approx \frac{2\pi\alpha^2}{s} \log\left(\frac{s}{m^2}\right)$$

Note this log is not small. Even worse for QCD, the collinear gluon emission factor is $\alpha_s^2(Q^2)\log\left(\frac{Q^2}{\mu^2}\right)$

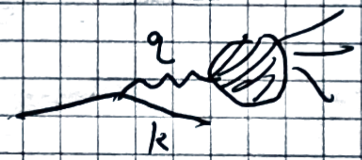
factorization scale.

The log enhancement is an IR divergence exactly analogous to FSR; in the initial state, we resolve the divergence by resumming the log into a photon PDF.

Consider the diagrams:



or



Technically, we will factorize on "almost" real polarization spinors or vectors for the intermediate e^- or photon. Essentially, can factorize ISR splitting from hard process' blob.

Study kinematics of collinear splitting:

$$p = (p, 0, 0, p)$$

$$q \approx \left(zp, p_\perp, 0, zp - \frac{p_\perp^2}{2zp}\right)$$

$$k \approx \left((1-z)p, -p_\perp, 0, (1-z)p + \frac{p_\perp^2}{2zp}\right)$$

using momentum conservation.

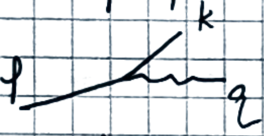
By keeping the photon real + electron slightly virtual, we have

$$k^2 = -\frac{p_\perp^2}{z} + \mathcal{O}(p_\perp^4) \quad \leftarrow \text{electron}$$

$$q^2 = 0 + \mathcal{O}(p_\perp^4) \quad \leftarrow \text{photon}$$

By symmetry, photon virtual $\quad q^2 = -\frac{p_\perp^2}{z}$

electron real $\quad k^2 = 0$



Now, the numerator of the virtual electron will be $\not{k} = \sum_s u^s(k)\,\bar{u}^s(k)$, where we factorize on the spins. Similarly, the photon is split up to $-\frac{ig^{\mu\nu}}{q^2} \to \frac{+i}{q^2}\sum \epsilon_{T_i}^{\mu}\,\epsilon_{T_i}^{\nu *}$, with an appropriate choice of ϵ_T, almost real polarization vectors.

We can then evaluate the explicit matrix element for electron splitting. It is convenient to break apart the matrix element into helicity components, since the collinear singularity behaves differently:

$$i\mathcal{M}(e_L^- \to e_L^- \gamma_R) = ie\frac{\sqrt{2(1-z)}}{z}\,p_\perp$$

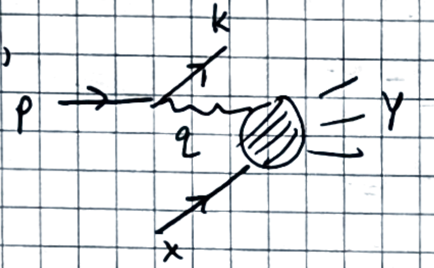
$$i\mathcal{M}(e_L^- \to e_L^- \gamma_L) = ie\frac{\sqrt{2(1-z)}}{z(1-z)}\,p_\perp$$

Same with $L \leftrightarrow R$.

We get $\quad \frac{1}{2}\sum_{\text{pols.}} |\mathcal{M}|^2 = \frac{2e^2 p_\perp^2}{z(1-z)}\left[\frac{1+(1-z)^2}{z}\right]$

This factor is a correction to the electron as a "parton."

Then,



Has the cross section

splitting $|M|^2$ from above

$$\sigma = \frac{1}{2p\, 2E_X}\frac{1}{(1+v_X)}\int\frac{d^3k}{(2\pi)^3}\frac{1}{2k^0}\int d\Pi_Y\left[\frac{1}{2}\Sigma|M|^2\right]\left(\frac{1}{q^2}\right)^2|M_{\gamma X}|^2$$

Change variables, $d^3k = dk^3\, d^2k_\perp = p\,dz\;\pi\, dp_\perp^2$

We get $$\sigma = \int\frac{p\,dz\,dp_\perp^2}{16\pi^2(1-z)p}\left[\frac{1}{2}\Sigma|M|^2\right]\frac{(1-z)^2}{p_\perp^4}\frac{z}{(1+v_X)(2zp)(2E_X)}\int d\Pi_Y|M_{\gamma X}|^2$$

$$= \int\frac{dz\,dp_\perp^2}{16\pi^2(1-z)}\left[\frac{1}{2}\Sigma|M|^2\right]\frac{z(1-z)^2}{p_\perp^4}\sigma(\gamma X\to Y)$$

Use $\frac{1}{2}\Sigma|M|^2 = \frac{2e^2p_\perp^2}{z(1-z)}\left[\frac{1+(1-z)^2}{z}\right]$

$$\sigma = \int\frac{dz\,dp_\perp^2}{16\pi^2(1-z)}\frac{2e^2p_\perp^2}{z(1-z)}\left(\frac{1+(1-z)^2}{z}\right)\frac{z(1-z)^2}{p_\perp^4}\sigma(\gamma X\to Y)$$

$$= \int_0^1 dz\int\frac{dp_\perp^2}{p_\perp^2}\frac{\alpha}{2\pi}\left(\frac{1+(1-z)^2}{z}\right)\sigma(\gamma X\to Y)$$

Similarly to Compton scattering, integral over $p_\perp$ is cut off by m_e,

$$\sigma(e^-X\to e^-Y) = \int_0^1 dz\,\frac{\alpha}{2\pi}\log\frac{s}{m^2}\left(\frac{1+(1-z)^2}{z}\right)\sigma(\gamma X\to Y)$$

We recognize the $\int_0^1 dz$ is exactly like an integral over a photon distribution fcn. with momentum fraction z.

$$f_\gamma(z) = \frac{\alpha}{2\pi}\log\frac{s}{m^2}\left(\frac{1+(1-z)^2}{z}\right)$$

$$\sigma(e^-X\to e^-Y) = \int_0^1 dz\; f_\gamma(z)\,\sigma(\gamma X\to Y)$$

This is called the ~~Williams~~ Weizsäcker-Williams equivalent photon approx.

By the same logic, collinear singularities for quarks are related to the gluon PDF.