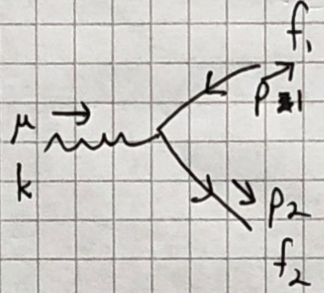
Lecture 2. Exercise.

Nov. 9, 2020.

Calculate Z and W branching fractions

Useful to practice calculations with SM tree level interactions.

We will calculate the general case with the vector decaying to $f_1 + f_2$ ~~p~~ fermions.



$$i\mathcal{M} = \bar{u}(p_1) \cdot i\,(g_V \gamma^\mu + g_A \gamma^\mu \gamma^5)\, v(p_2) \cdot \epsilon_\mu(k)$$

$$|\mathcal{M}|^2 = \mathrm{Tr}\left[(g_V \gamma^\mu + g_A \gamma^\mu \gamma^5)(\not{p}_2 - m_2)(g_V \gamma^\nu + g_A \gamma^\nu \gamma^5)(\not{p}_1 + m_1) \right] \cdot \epsilon_\mu(k)\, \epsilon^*_\nu(k)$$

$$\sum_s v\bar{v} = \not{p}_2 - m_2 \mathbb{1}$$

$$\sum_s u\bar{u} = \not{p}_1 + m_1 \mathbb{1}$$

Multiply out trace:

$$= \epsilon_\mu \epsilon^*_\nu \, \mathrm{Tr}\big[g_V^2 \gamma^\mu (\not{p}_2 - m_2) \gamma^\nu (\not{p}_1 + m_1)$$
$$+ g_V g_A \gamma^\mu \gamma^5 (\not{p}_2 - m_2) \gamma^\nu (\not{p}_1 + m_1)$$
$$+ g_V g_A \gamma^\mu (\not{p}_2 - m_2) \gamma^\nu \gamma^5 (\not{p}_1 + m_1)$$
$$+ g_A^2 \gamma^\mu \gamma^5 (\not{p}_2 - m_2) \gamma^\nu \gamma^5 (\not{p}_1 + m_1) \big]$$

We know that $\sum_{\text{pols.}} \epsilon_\mu \epsilon^*_\nu$ is symmetric in $\mu \leftrightarrow \nu$, since $\sum_{\text{pols.}} \epsilon_\mu \epsilon^*_\nu = -g_{\mu\nu} + \frac{k_\mu k_\nu}{m_V^2}$. So, the terms with one γ^5 vanish.

$$= \epsilon_\mu \epsilon^*_\nu \Big(g_V^2 \big(4 (p_2^\mu p_1^\nu - p_1 \cdot p_2\, g^{\mu\nu} + p_2^\nu p_1^\mu) - 4 g^{\mu\nu} m_1 m_2 \big)$$
$$+ g_A^2 \big(4 (p_2^\mu p_1^\nu - p_1 \cdot p_2\, g^{\mu\nu} + p_2^\nu p_1^\mu) + 4 g^{\mu\nu} m_1 m_2 \big) \Big)$$

$$|\mathcal{M}|^2 = 4\epsilon_\mu \epsilon_\nu^* \Big[(g_V^2+g_A^2)\left(p_2^\mu p_1^\nu - p_1 \cdot p_2 g^{\mu\nu} + p_2^\nu p_1^\mu\right) - (g_V^2-g_A^2)\left(g^{\mu\nu} m_1 m_2\right)\Big]$$

$$\sum_{\text{pols.}} |\mathcal{M}|^2 = 4\left(-g_{\mu\nu} + \frac{k_\mu k_\nu}{m_V^2}\right)\Big[(g_V^2+g_A^2)\left(p_2^\mu p_1^\nu - p_1 \cdot p_2 g^{\mu\nu} + p_2^\nu p_1^\mu\right) - (g_V^2-g_A^2) m_1 m_2 g^{\mu\nu}\Big]$$

$$= 4\Big[\left(2p_1\cdot p_2 + 2\frac{k\cdot p_2\, k\cdot p_1}{m_V^2} - \frac{p_1\cdot p_2\, k^2}{m_V^2}\right)(g_V^2+g_A^2) + (g_V^2-g_A^2)m_1 m_2\left(4 - \frac{k\cdot k}{m_V^2}\right)\Big]$$

$$k\cdot p_1 = (p_1+p_2)\cdot p_1 = m_1^2 + p_1\cdot p_2 = m_1^2 + \tfrac{1}{2}\left((p_1+p_2)^2 - p_1^2 - p_2^2\right)$$
$$= m_1^2 + \tfrac{1}{2}\left(m_V^2 - m_1^2 - m_2^2\right)$$
$$= \tfrac{1}{2}\left(m_V^2 + m_1^2 - m_2^2\right)$$

Also, $k\cdot p_2 = \frac{1}{2}\left(m_V^2 - m_1^2 + m_2^2\right)$

$$\sum_{\text{pols.}} |\mathcal{M}|^2 = 4\Big[\left(p_1\cdot p_2 + \frac{2}{m_V^2}\left(\frac{1}{4}\right)(m_V^2+m_1^2-m_2^2)(m_V^2-m_1^2+m_2^2)\right)(g_V^2+g_A^2) + 3(g_V^2-g_A^2)m_1 m_2\Big]$$

$$= 4\Big[\left(\tfrac{1}{2}(m_V^2-m_1^2-m_2^2) + \tfrac{1}{2}\left(1+\frac{m_1^2-m_2^2}{m_V^2}\right)(m_V^2-m_1^2+m_2^2)\right)(g_V^2+g_A^2) + 3(g_V^2-g_A^2)m_1 m_2\Big]$$

$$= 4\Big[\tfrac{1}{2}\left(m_V^2-m_1^2-m_2^2+m_V^2-\cancel{m_1^2}+\cancel{m_2^2}+\cancel{m_1^2}-\cancel{m_2^2}-\frac{(m_1^2-m_2^2)^2}{m_V^2}\right)(g_V^2+g_A^2) + 3(g_V^2-g_A^2)m_1 m_2\Big]$$

The $\sum_{\text{pols.}}|\mathcal{M}|^2$ is only dependent on Lorentz scalar invariants - can evaluate phase space integral separately.

$$d\Gamma = \frac{1}{2m_V}|\mathcal{M}|^2 \cdot \int \frac{d\Omega}{4\pi}\,\frac{1}{8\pi}\,\frac{2|\vec{p}|}{E_{CM}}$$

$$f_2 \longleftarrow \times \longrightarrow f_1$$

$$p_{f_1} = (E_{f_1}, \vec{p}) \qquad E_{f_1} = \sqrt{m_{f_1}^2 + |\vec{p}|^2}$$
$$p_{f_2} = (E_{f_2}, -\vec{p}) \qquad E_{f_2} = \sqrt{m_{f_2}^2 + |\vec{p}|^2}$$
$$E_{f_1} + E_{f_2} = m_V$$

$$\Rightarrow |\vec{p}|^2 = \frac{1}{4m_V^2}\left(m_V^4 - 2m_V^2 m_{f_1}^2 - 2m_V^2 m_{f_2}^2 - 2m_{f_1}^2 m_{f_2}^2 + m_{f_1}^4 + m_{f_2}^4\right)$$

This very common phase space expression is the Kallen λ function:

$$\text{Kallen}\lambda\ (x,y,z) \equiv x^2+y^2+z^2-2xy-2yz-2zx.$$

$$|\vec{p}|^2 = \frac{1}{4m_V^2}\,\lambda(m_V^2, m_{f_1}^2, m_{f_2}^2)$$

$$\Gamma = \frac{1}{2m_V}\cdot\frac{1}{8\pi}\cdot\frac{2}{m_V}\cdot\frac{1}{2m_V}\cdot\sqrt{\lambda(m_V^2,m_{f_1}^2,m_{f_2}^2)}\cdot|M|^2$$

$$= \frac{1}{16\pi}\frac{1}{m_V^3}\sqrt{\lambda(m_V^2,m_{f_1}^2,m_{f_2}^2)}$$

$$\cdot 4\left((g_V^2+g_A^2)\tfrac{1}{2}\left(2m_V^2-m_1^2-m_2^2-\frac{(m_1^2-m_2^2)^2}{m_V^2}\right) + (g_V^2-g_A^2)\,3\,m_1m_2\right)$$

Note that the sum over final state fermions requires an N_c factor when we sum over quarks.

Also, we must average over vector polarizations: factor of $\frac{1}{3}$.

$$\Gamma(V\to\bar{f}_1 f_2) = \frac{1}{12\pi}\frac{1}{m_V^3}\sqrt{\lambda(m_V^2,m_1^2,m_2^2)}\;N_c$$

$$\left(\tfrac{1}{2}(g_V^2+g_A^2)\left(2m_V^2-m_1^2-m_2^2-\frac{(m_1^2-m_2^2)^2}{m_V^2}\right) + 3(g_V^2-g_A^2)\,m_1m_2\right)$$