Lecture 13, January 13, 2021

Wednesday, January 13, 2021 2:15 PM

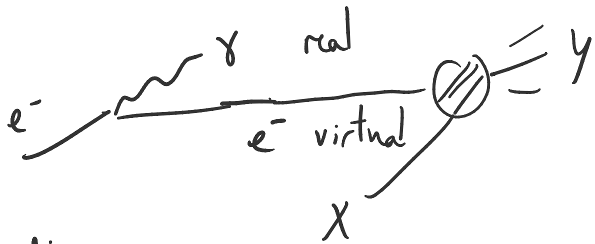
Today: finish $e^- \to \gamma$ PDFs

Last time,

we had derived the Weizsäcker-Williams equivalent photon approximation.

Recall, e^- incoming: Analyze collinear photon emission:

virtual pols. ~ on-shell up to $\mathcal{O}(p_\perp^2)$



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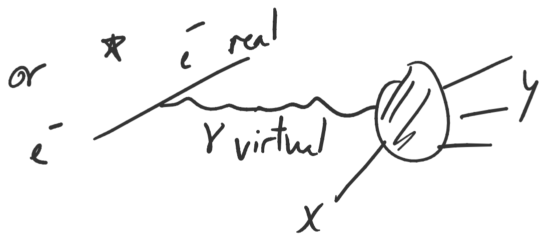


For example: * RH diagram:

$\epsilon_L^{*i}(q) = \frac{1}{\sqrt{2}}\left(1, i, -\frac{p_\perp}{zp}\right)$

$\epsilon_R^{*i}(q) = \frac{1}{\sqrt{2}}\left(1, -i, -\frac{p_\perp}{zp}\right)$

$$\sigma(e^- X \to e^- Y) = \int_0^1 dz\, \frac{\alpha}{2\pi} \log\left(\frac{s}{m^2}\right)\left(\frac{1+(1-z)^2}{z}\right)\sigma(\gamma X \to Y)$$

$$f_\gamma(z) = \frac{\alpha}{2\pi} \log\frac{s}{m^2}\left(\frac{1+(1-z)^2}{z}\right), \text{ photon PDF}$$

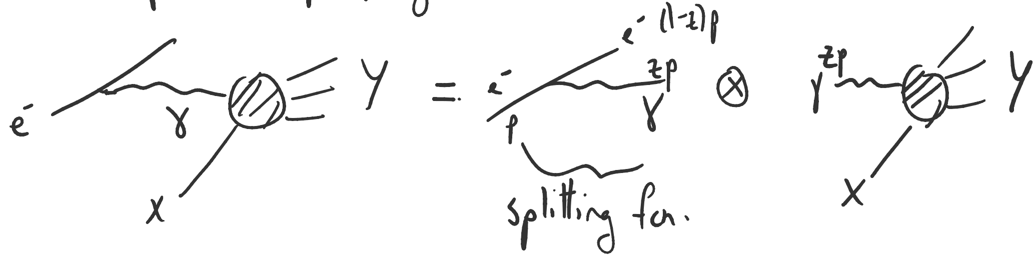
$\sigma(\gamma X \to Y)$: "hard" cross section for γ with momentum zp.

z : fraction of momentum carried by photon.

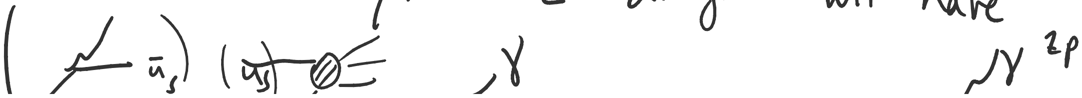
$$\sigma(e^- X \to e^- Y) = \int_0^1 dz\, f_\gamma(z)\, \sigma(\gamma(zp)\, X \to Y)$$

This is a subprocess for the total Xsec.

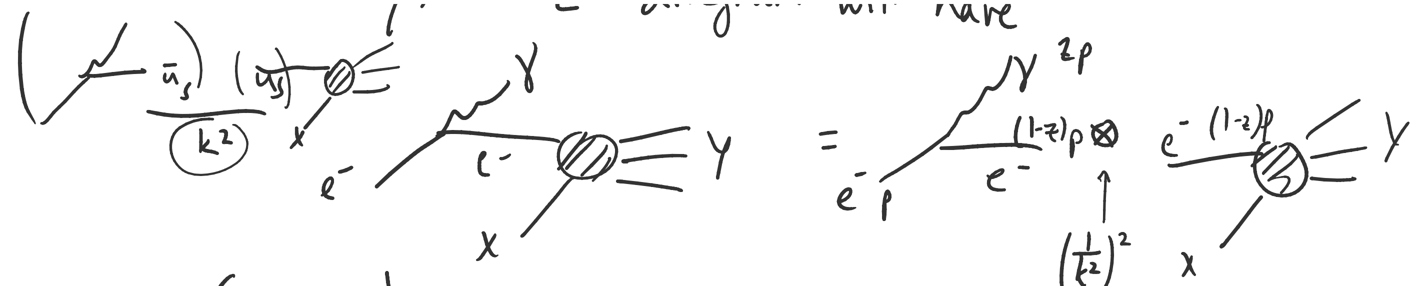
Recall, mathematically, we factorized hard process on the virtual photon propagator, s.t. total xsec became



Similarly, the LH diagram will have



diagrams will have



So, second process:

$$\sigma = \frac{1}{(1+v_X)\,2p\,2E_X}\int \frac{d^3q}{(2\pi)^3}\frac{1}{2q^0}\int d\Pi_Y \left[\frac{1}{2}\sum_s |\mathcal{M}|^2\right]\left(\frac{1}{k^2}\right)^2 |\mathcal{M}_{e^-X}|^2$$

e^- has $(1-z)p$ momentum

Follow same analysis:

$$\sigma(e^-X\to\gamma Y) = \int \frac{dz}{16\pi^2}\frac{dp_\perp^2}{z}\left[\frac{1}{2}\sum_s |\mathcal{M}|^2\right]\frac{z^2}{p_\perp^4}(1-z)\,\sigma(e^-X\to Y)$$

$$= \int \frac{dz}{16\pi^2}\,dp_\perp^2\,\frac{z(1-z)}{p_\perp^4}\,\frac{2e^2 p_\perp^2}{z(1-z)}\left[\frac{1+(1-z)^2}{z}\right]\sigma(e^-X\to Y)$$

$$= \int_0^1 dz\int \frac{dp_\perp^2}{p_\perp^2}\,\frac{\alpha}{2\pi}\left(\frac{1+(1-z)^2}{z}\right)\sigma(e^-(1-z)_p\, X\to Y)$$

We recognize $f_e^{(1)}(x) = \frac{\alpha}{2\pi}\log\frac{s}{m^2}\left(\frac{1+x^2}{1-x}\right)$ for $x\equiv 1-z$

& $\int \frac{dp_\perp^2}{p_\perp^2}$ is the collinear IR div. (as in Compton scattering) cut off by m_e^2:

$f_e^{(1)}(x)$ is essentially the $\mathcal{O}(\alpha)$ PDF for the e^- parton from e^-. Need to include $\mathcal{O}(\alpha^0)$ PDF, which had no splitting.

The $f_e^{(0)}(x) = \delta(1-x)$.

$$\frac{d\sigma_{tot}}{dz} = \frac{d\sigma}{dz}(e^-(p)X\to Y) + \frac{d\sigma}{dz}(e^-(xp)X\to Y)\cdot\text{split}\,(e^-(p)\to e^-(xp)\,\gamma[(1-x)p])$$
$$+\frac{d\sigma}{dz}(\gamma(zp)X\to Y)\cdot\text{split}(e^-(p)\to e^-(xp)\,\gamma(zp))$$

$$dt \qquad dz \qquad + \frac{d\sigma}{dz}(\gamma(zp)\, X \to Y) \cdot \text{split}(e(p) \to e(xp) + \gamma(\ldots p))$$

the appropriate $f^{(0)}_e(x)$ for first term is $\delta(1-x)$.

To $O(\alpha)$, PDF of e^- is

$$f_e(x) = \delta(1-x) + \frac{\alpha}{2\pi} \log\left(\frac{s}{m^2}\right)\left(\frac{1+x^2}{1-x} - A\,\delta(1-x)\right)$$

s.t.

$$\underset{\text{total}}{\sigma} = \int_0^1 dx \sum_i f_i(x)\, \underset{\text{partonic xsec.}}{\hat{\sigma}}(i\, X \to Y)$$

Why introduce A? Essentially conserve probability from moving radiation from $x=1$ to finite $x<1$ @ $O(\alpha)$.

If we ignore pair creation, fix A by $\int_0^1 dx\, f_e(x) = 1$.

Problem: how to perform integration of singular denominator?

$\frac{1}{1-x}$ denominator corresponds to emission of soft photons.

Mathematically, use subtraction procedure:

$\frac{1}{(1-x)_+} \equiv \frac{1}{1-x}$ for $x<1$, & for $x=1$, matches singularity at $x=1$.

In practice, $\int_0^1 dx \frac{f(x)}{(1-x)_+} = \int_0^1 dx \frac{f(x) - f(1)}{1-x}$ ← IR regulator

LHS is now manifestly non-singular.

same role: $\mathcal{L}(\lambda_b, \alpha_b, \ldots) - \mathcal{L}_{ct}(\delta\lambda, \delta_\alpha) = \mathcal{L}_{PT}$ finite

This is reminiscent of counterterms in UV divergences from RG.

Perform replacement:

$$f_e(x) = \delta(1-x) + \frac{\alpha}{2\pi} \log\frac{s}{m^2}\left(\frac{1+x^2}{(1-x)_+} - A\,\delta(1-x)\right)$$

Get A from $\int_0^1 dx\, f_e(x) = 1 \Rightarrow \int_0^1 dx \frac{1+x^2}{(1-x)_+} =$

... N from $\int_0^1 dx\, f_e(x) = 1 \Rightarrow \int_0^1 dx \frac{1+x^2}{(1-x)_+} =$
$\int_0^1 dx \frac{1+x^2-2}{(1-x)} = -\frac{3}{2}$

$$f_e(x) = \delta(1-x) + \frac{\alpha}{2\pi}\log\left(\frac{s}{m^2}\right)\left(\frac{1+x^2}{(1-x)_+} + \frac{3}{2}\delta(1-x)\right)$$

Come back @ 3:22pm.

This was the $O(\alpha)$ effect on an incoming e^- beam.
Splits into e^- PDF (parton from initial e^-)
and γ PDF (parton from initial e^-).

Multiple splittings:

For $p_{\perp,2} \ll p_{\perp,1}$ the virtuality of e^- leg is unchanged & we get a compounding collinear enhancement:

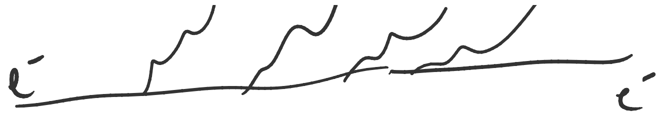
$$\frac{1}{2!}\left(\frac{\alpha}{2\pi}\right)^2 \log^2\left(\frac{s}{m^2}\right) \left(= \left(\frac{\alpha}{2\pi}\right)^2 \int_{m^2}^{s} \frac{dp_{\perp,1}^2}{p_{\perp,1}^2} \int_{m^2}^{p_{\perp,1}^2} \frac{dp_{\perp,2}^2}{p_{\perp,2}^2}\right)$$

For n collinear emissions,

$\frac{1}{n!}\left(\frac{\alpha}{2\pi}\right)^n \log^n\left(\frac{s}{m^2}\right)$, which should be resummed, since $\frac{\alpha}{2\pi}\log\frac{s}{m^2} \sim 1$

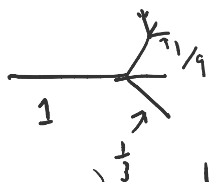
Note : $p_{\perp,2} \gg p_{\perp,1}$, there is no double log

Only for $p_{\perp j} \ll p_{\perp j-1}$ do you get compounding enhancement $\exists$ "strongly ordered".



We can intuit that we're probing e^- line at successively smaller distance scales.

Adopt usual procedure of treating the splitting function as a kernel for Q^2 evolution.



For next step / iteration I repeat the rule.

↳ kernel = mathematical rule

step/iteration ⇒ continuum = differential in log.

$$\left[\frac{d}{dx} f(x) = K(x) \underbrace{f(x)}\right]$$

$K(x) = 1,$
$\Rightarrow f(x) = e^x$

$f(x) = e^x \Rightarrow$ self-similarity that is trivial. $\underbrace{\frac{d}{dx}} f(x) = f(x)$

For $K(x) = 2,$

☆ $f(x) = \frac{e^{K(x)x}}{|K(x)|}$

$$\underbrace{\frac{d}{d\log Q}} f_\gamma(x, Q) = \int_x^1 \frac{dz}{z} \left(\frac{\alpha}{\pi} \frac{1 + (1-z)^2}{z} \right) f_e\left(\frac{x}{z}, Q\right)$$

↑ kernel ↑ seed = boundary condition

$$\underbrace{\frac{d}{d\log Q}} f_e(x, Q) = \int_x^1 \frac{dz}{z} \left(\frac{\alpha}{\pi} \left(\frac{1+z^2}{(1-z)_+} + \frac{3}{2}\delta(1-z) \right) \right) \underbrace{f_e\left(\frac{x}{z}, Q\right)}$$

☆ One step before:

Consider $f_e(x, Q)$ + $f_\gamma(x, Q)$.

Consider how they are related / inherited from

$f_e(x, Q + \Delta Q)$ + $f_\gamma(x, Q + \Delta Q)$.

$f_\gamma(x, Q + \Delta Q) = f_\gamma(x, Q)$ + split arising from f_e at $Q + \Delta Q$ losing exactly photon of ΔQ.

[at $O(\alpha)$].

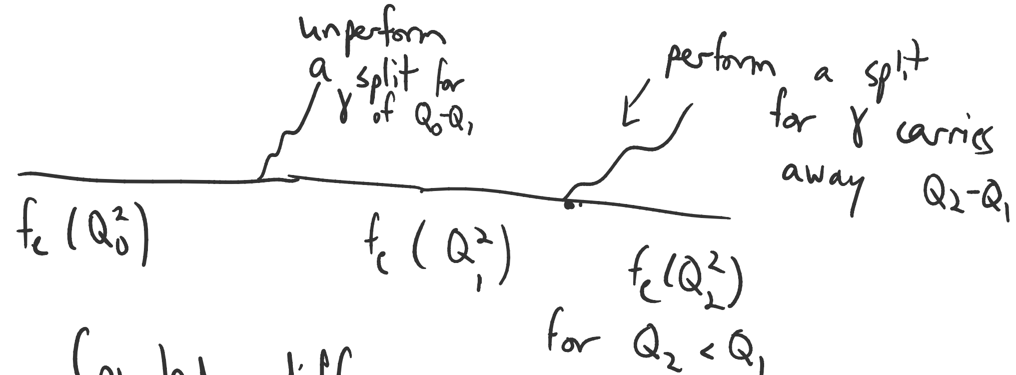
x is the momentum fraction of ...

[at $O(\alpha)$].

x is the momentum fraction of parent e^- for PDF $f_e(x)$

Q is a momentum transfer observable, $Q > p_\perp$.

Move away from beam of fixed energy, & instead understand differences of $e^- + \gamma$ PDFs relative to each other.



Coupled diff. eqn.

$$\frac{d}{d\log Q} f_i = \int_x^1 \frac{dz}{z} \underset{\substack{\uparrow \\ \text{kernel}}}{M_{ij}} f_j\left(\frac{x}{z}, Q^2\right)$$

When we include $\gamma \to e^+e^-$ splitting, then we get complete set of PDF evolution equations that resum collinear singularities to all orders in α, leading log.

$$\frac{d}{d\log Q} f_\gamma(x,Q) = \frac{\alpha}{\pi}\int_x^1 \frac{dz}{z}\left[P_{\gamma\leftarrow e}(z)\left[f_e + f_{\bar{e}}\right) + P_{\gamma\leftarrow\gamma}(z) f_\gamma\right]$$

Gribov + Lipatov
$$\frac{d}{d\log Q} f_e(x,Q) = \frac{\alpha}{\pi}\int_x^1 \frac{dz}{z}\left[P_{e\leftarrow e}(z) f_e + P_{e\leftarrow\gamma} f_\gamma\right]$$

$$\frac{d}{d\log Q} f_{\bar{e}} = \frac{\alpha}{\pi}\int_x^1 \frac{dz}{z}\left[P_{e\leftarrow e} f_{\bar{e}} + P_{e\leftarrow\gamma} f_\gamma\right]$$

$$P_{e\leftarrow e}(z) = \frac{1+z^2}{\dots} + \frac{3}{2}\delta(1-z)$$

$$P_{e\leftarrow e}(z) = \frac{1+z^2}{(1-z)_+} + \frac{3}{2}\delta(1-z)$$

$$P_{\gamma\leftarrow e}(z) = \frac{1+(1-z)^2}{z}$$

$$P_{e\leftarrow \gamma}(z) = z^2+(1-z)^2$$

$$P_{\gamma\leftarrow \gamma}(z) = -\frac{2}{3}\delta(1-z)$$