Renormalizing $\lambda\phi^4$ theory & anomalous dimension scaling. ①

Quantum Field Theory II.

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In this note, we renormalize $\lambda\phi^4$ theory and demonstrate that the anomalous dimension term begins at $\mathcal{O}(\lambda^2)$.

bare $\mathcal{L} = \frac{1}{2}(\partial_\mu\phi)^2 - \frac{1}{2}m_0^2\phi^2 - \frac{\lambda_0}{4!}\phi^4$

perform field strength renormalization & counterterms

$$\phi = Z^{1/2}\phi_r,$$

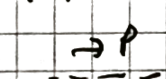
$$\delta_Z = Z-1, \quad \delta_m = m_0^2 Z - m^2, \quad \delta_\lambda = \lambda_0 Z^2 - \lambda$$

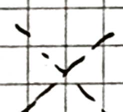
$$\mathcal{L} = \frac{1}{2}Z(\partial_\mu\phi_r)^2 - \frac{1}{2}m_0^2 Z\phi_r^2 - \frac{\lambda_0}{4!}Z^2\phi_r^4$$

$$= \frac{1}{2}(\partial_\mu\phi_r)^2 - \frac{1}{2}m^2\phi_r^2 - \frac{\lambda}{4!}\phi_r^4$$

$$+ \frac{\delta_Z}{2}(\partial_\mu\phi_r)^2 - \frac{1}{2}\delta_m\phi_r^2 - \frac{\delta_\lambda}{4!}\phi_r^4$$

Feynman rules:

 $\frac{i}{p^2-m^2+i\epsilon}$

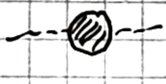
 $-i\lambda$

 $i(p^2\delta_Z - \delta_m)$

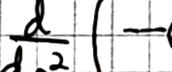
 $-i\delta_\lambda$

Renormalization conditions.

2-pt.

 $= \dfrac{i}{p^2 - m^2 + \Pi(p^2)}$

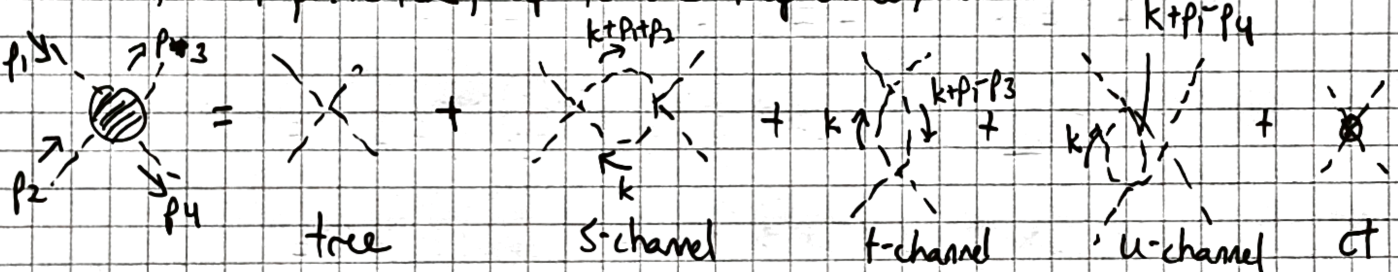
where $\Pi(p^2)\big|_{p^2=m^2} = 0$ and the residue is a

simple pole at $p^2 = m^2$. $\Rightarrow$ $\dfrac{d}{dp^2}\Big($  $\Big) = 0$ (eqn. 10.28)

4-pt.  $= -i\lambda$ at $s = t = u = -\mu^2$

We calculate the counterterms by imposing the ren. conditions

The 4-pt. vertex, up to one-loop order, is



$$i\mathcal{M}_{tree} = -i\lambda$$

$$i\mathcal{M}_{ct} = -i\delta_\lambda$$

$$i\mathcal{M}_{s\text{-channel}} = \frac{(-i\lambda)^2}{2} \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2-m^2} \frac{i}{(k+p_1+p_2)^2 - m^2}$$

$$i\mathcal{M}_{t\text{-channel}} = \frac{(-i\lambda)^2}{2} \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2-m^2} \frac{i}{(k+p_1-p_3)^2 - m^2}$$

$$i\mathcal{M}_{u\text{-channel}} = \frac{(-i\lambda)^2}{2} \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2-m^2} \frac{i}{(k+p_1-p_4)^2 - m^2}$$

Following Peskin, define $iV(p^2) = \dfrac{1}{2} \displaystyle\int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2-m^2} \frac{i}{(k+p)^2 - m^2}$

$$iV(p^2) = -\frac{1}{2}\int \frac{d^dk}{(2\pi)^d}\int_0^1 dx \frac{1}{(k^2-m^2+2k\cdot p x + xp^2)^2}$$

$$= -\frac{1}{2}\int \frac{d^dk}{(2\pi)^d} \frac{1}{((k-xp)^2 - m^2 + xp^2 - x^2p^2)^2}$$

$$\ell \equiv k - xp, \quad \Delta \equiv m^2 + x(x-1)p^2$$

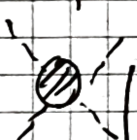
$$= -\frac{1}{2}\int \frac{d^d\ell}{(2\pi)^d}\int dx \frac{1}{(\ell^2-\Delta)^2}$$

$$\int \frac{d^d\ell}{(2\pi)^d} \frac{1}{(\ell^2-\Delta)^n} = \frac{(-1)^n i}{(4\pi)^{d/2}} \cdot \frac{\Gamma(n-\frac{d}{2})}{\Gamma(n)} \left(\frac{1}{\Delta}\right)^{n-\frac{d}{2}}$$

$$iV(p^2) = -\frac{1}{2}\cdot \int dx \frac{i}{(4\pi)^{d/2}} \cdot \frac{\Gamma(2-\frac{d}{2})}{(\Delta)^{2-d/2}}$$

$$iV(p^2) = \frac{-i}{2(16\pi^2)} \int dx \left(\frac{2}{\epsilon} - \gamma + \log 4\pi - \log \Delta\right)$$

Given, at one-loop, the sum of all diagrams is simply $-i\lambda$, we have the CT cancelling the loop diagrams for $s=t=u=-M^2$:

$-i\lambda =$  $\Big|_{s=t=u=-M^2} = \underbrace{-i\lambda}_{\text{tree}} + \left(\underset{\text{channel}}{s+t+u} + CT\right)\Big|_{s=t=u=-M^2}$

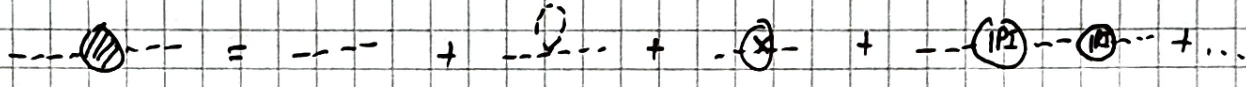
$$\Rightarrow +i\delta_\lambda = (-i\lambda)^2 \cdot \left(iV(s) + iV(t) + iV(u)\right)\Big|_{s=t=u=-M^2}$$

$$i\delta_\lambda = \frac{-i\lambda^2}{2(16\pi^2)}\left(\int dx \frac{6}{\epsilon} - 3\gamma + 3\log 4\pi - \log\Delta_s - \log\Delta_t - \log\Delta_u\right)$$

with $\log \Delta_s = \log\left(m^2 + x(x-1)(-M^2)\right) \equiv \log \Delta_{M^2}$

$$i\delta_\lambda = \frac{-i3\lambda^2}{2(16\pi^2)} \int dx \left(\frac{2}{\epsilon} - \gamma + \log 4\pi - \log\Delta_{M^2}\right)$$

We calculate the propagator & mass correction:



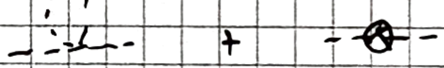
$$= \frac{i}{p^2 - m^2 - M^2(p^2)} \quad \text{where} \quad 1 + x + x^2 + \ldots = \frac{1}{1-x}$$

Define --(1PI)-- $= -iM^2(p^2)$

Geometrically: $\dfrac{i}{p^2-m^2} + \dfrac{i}{p^2-m^2}(-iM^2)\dfrac{i}{p^2-m^2} + \dfrac{i}{p^2-m^2}(-iM^2)\dfrac{i}{p^2-m^2}(-iM^2)\dfrac{i}{p^2-m^2} + \ldots$

$$= \frac{i}{p^2-m^2}\left(1 + \frac{M^2}{p^2-m^2} + \left(\frac{M^2}{p^2-m^2}\right)^2 + \ldots\right)$$

$$= \frac{i}{p^2-m^2}\left(\frac{1}{1-\left(\frac{M^2}{p^2-m^2}\right)}\right) = \frac{i}{p^2-m^2-M^2(p^2)} \quad \checkmark$$

So $-iM^2(p^2) =$ 

$$= -i\lambda \cdot \frac{1}{2}\int \frac{d^dk}{(2\pi)^d}\frac{i}{k^2-m^2} + i(p^2\delta_z - \delta_m)$$

$$= \frac{-i\lambda}{2} \cdot \frac{\cancel{i}(-1)\cancel{i}}{16\pi^2} \cdot \frac{\Gamma(1-\frac{d}{2})}{\Delta^{1-d/2}} + i(p^2\delta_z - \delta_m)$$

$$-iM^2(p^2) = \frac{-i\lambda}{32\pi^2}\frac{\Gamma(1-d/2)}{\Delta^{1-d/2}} + i(p^2\delta_z - \delta_m)$$

Evaluate at $p^2 = m^2$, $\Delta = m^2$.

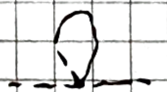
$M^2(p^2=m^2)=0$

$\left.\dfrac{d}{dp^2}M^2(p^2)\right|_{p^2=m^2} = 0$

$$\begin{cases} -iM^2(p^2=m^2) = \dfrac{-i\lambda}{32\pi^2}\dfrac{\Gamma(1-\frac{d}{2})}{(m^2)^{1-d/2}} + i(m^2\delta_z - \delta_m) = 0 \\ \left.-i\dfrac{dM^2}{dp^2}\right|_{p^2=m^2} = 0 + i\delta_z = 0 \end{cases}$$

$$\Rightarrow \delta_z = 0 \quad \text{and} \quad \delta_m = \frac{-\lambda}{32\pi^2}\frac{\Gamma(1-d/2)}{(m^2)^{1-d/2}}$$

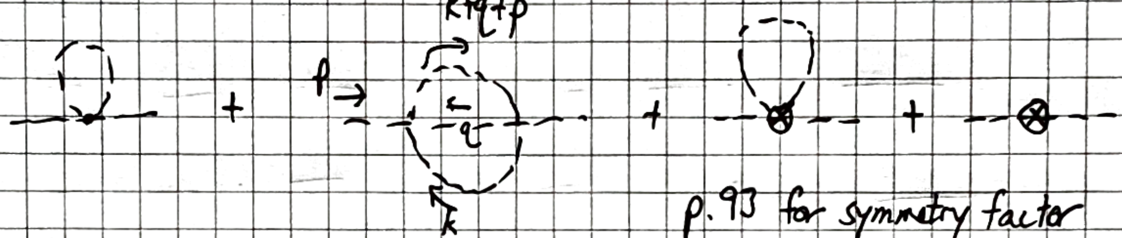
Notice that the $\delta_Z = 0$ is a consequence of

 being a pure constant shift with no

momentum dependence. This is the reason that the anomalous dimension is only first seen at $\mathcal{O}(\lambda^2)$.

We have $\left[M \frac{\partial}{\partial M} + \beta(\lambda) \frac{\partial}{\partial \lambda} + 2\gamma(\lambda) \right] G^{(2)}(p) = 0$

At one- and two-loops we have 1PI:



p. 93 for symmetry factor

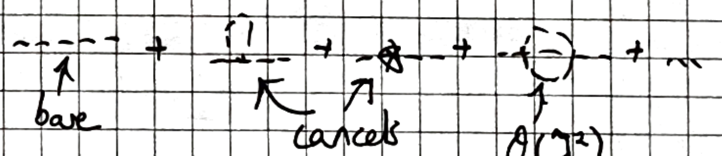
$$-iM^2(p^2) = \frac{-i\lambda}{32\pi^2} \frac{\Gamma(1-d/2)}{(m^2)^{1-d/2}} + \frac{(-i\lambda)^2}{6} \int \frac{d^4k}{(2\pi)^4} \int \frac{d^4q}{(2\pi)^4} \frac{i}{k^2-m^2} \frac{i}{q^2-m^2} \frac{i}{(k+q+p)^2-m^2}$$

$$- \frac{i\delta_\lambda}{2} \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2-m^2} + i(p^2\delta_Z - \delta_m)$$

$i\delta_\lambda$ is derived from 1-loop, as before. δ_Z is now non-zero and is $\mathcal{O}(\lambda^2)$.

Order by order in λ, we have redefinitions of δ_m, so first term is already taken care of.

Given that $G^{(2)}(p) =$



we have leading anomalous dimension at $\mathcal{O}(\lambda^2)$.