Lecture 8.

QFT II.

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Last time: Finished renormalization, asymptotic behavior

Today: Path integrals + generating functionals

Final comments about renormalization

A central concept about renormalization is the appearance of a scale in the theory where the Lagrangian has only dimensionless couplings. Fundamentally, this arises because of the Weyl or scale anomaly (see Shifman, Advanced Topics in QFT, Chap. 36), but intuitively, this phenomenon occurs because renormalization conditions define a scale to set the counterterms. Equivalently, the RG equation requires a scale to solve the ODE given by the Callan-Symanzik equation. This phenomenon is known as dimensional transmutation, and is one of the central topics explored in Homework 3.

A particularly instructive example of dimensional transmutation occurs in scalar QED when the quartic self-interaction of the scalar field is small. This is known as the Coleman-Weinberg model, & the effective potential (which is calculated from resumming the 1-loop diagrams with $2n$ external legs, $n \in \mathbb{N}$) leads to a vacuum expectation value for the scalar field. This is known as the Higgs mechanism; QED is spontaneously broken and the photon becomes massive.

We now turn our attention to the second main unit of this course: path integrals and functional methods.

The path integral formalism gives an alternate approach to calculating correlation fcns.

First, revisit quantum mechanics and path integrals.

Mainly focused in QM on the amplitude for particle to travel from x_a to x_b in time T:

$$U(x_a, x_b; T) = \langle x_b | e^{-iHT/\hbar} | x_a \rangle$$

Recall that we had two equivalent pictures:

Schrödinger: bra + ket get time evolution, operators fixed.

Heisenberg: operators evolve in time, states fixed.

Can formulate as x_a going to x_b over all paths, with each path contributing some phase

$$U(x_a, x_b; T) = \int \mathcal{D}x(t)\, e^{i(\text{phase})}$$

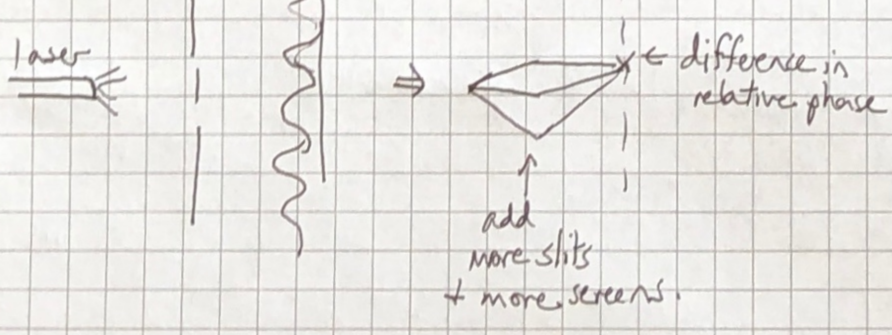
The phase is the action, $S = \int L\, dt = \int d^4x\, \mathcal{L}$, where classical path $(\hbar \to 0)$ is from the method of stationary phase.

$$\langle x_b | e^{-iHT/\hbar} | x_a \rangle = \int \mathcal{D}x(t)\, e^{iS[x(t)]/\hbar}$$

Functional: map functions to numbers

Integral occurs over function space, using measure $\mathcal{D}$.

Pictorial motivation for path integral: Continuum limit of infinite double slits. Essentially, must sum over phases w/ coherent interference.



Sketch procedure

Transition from coordinate q_a to q_b.

Coordinates: q^i, conjugate momenta: p^i

Hamiltonian: $H(\underline{q}, \underline{p})$ $\qquad q_a = \{q_a^i\}$

$U(q_a, q_b; T) = \langle q_b | e^{-iHT} | q_a \rangle$ $\qquad$ All coordinates

1. Split T into N slices: $e^{-iH\epsilon} \ldots e^{-iH\epsilon}$
2. Insert complete set of coordinate eigenstates.

$$\mathbb{1} = \int \prod_a dq_a |\underline{q}\rangle\langle \underline{q}|$$

$$\mathbb{1} = \left(\int \prod_i dq_k^i\right) |q_k\rangle\langle q_k| \qquad \mathbb{1} = \left(\prod_i \int dp_k^i\right) |p_k\rangle\langle p_k|$$

Repeat $N-1$ times:

Get $\langle q_{k+1} | e^{-iH\epsilon} | q_k \rangle \rightarrow \langle q_{k+1} | 1 - iH\epsilon + \ldots | q_k \rangle$

3. Operate H on eigenstates. Pure coordinates easy.

Consider: $\langle q_{k+1} | f(q) | q_k \rangle = f(q_k) \prod_i \delta(q_k^i - q_{k+1}^i)$

or $= f\left(\frac{q_{k+1} + q_k}{2}\right) \left(\prod_i \int \frac{dp_k^i}{2\pi}\right) \exp\left(i \sum_i p_k^i (q_{k+1}^i - q_k^i)\right)$

Also, $\langle q_{k+1} | f(p) | q_k \rangle$

$$= \left(\prod_i \int \frac{dp_k^i}{2\pi}\right) f(p_k) \exp\left[i \sum_i p_k^i (q_{k+1}^i - q_k^i)\right]$$

For Weyl-ordering, with all conjugate momenta to left of coordinates, or from commuting p & q, can symmetrize q on left and right.

$$\langle q_{k+1} | e^{-i\epsilon H} | q_k \rangle = \left(\prod_i \int \frac{dp_k^i}{(2\pi)}\right) \exp\left[-i\epsilon H\left(\frac{q_{k+1} + q_k}{2}, p_k\right)\right]$$

$$\exp\left[i \sum_i p_k^i (q_{k+1}^i - q_k^i)\right]$$

We have N such inner products

$$U(q_0, q_N; T) = \left(\prod_{i,k} \int dq_k^i \int \frac{dp_k^i}{(2\pi)}\right) \exp\left[i \sum_k \left(\sum_i p_k^i (q_{k+1}^i - q_k^i) - \epsilon H\left(\frac{q_{k+1} - q_k}{2}, p_k\right)\right)\right]$$

$N \to \infty$

$$= \left(\prod_i \int Dq(t) Dp(t)\right) \exp\left[i \int_0^T dt \left(\sum_i p^i \dot{q}^i - H(q, p)\right)\right]$$

$q(t)$ constrained by endpoints $(q(t=0) = q_0,\ q(t=T) = q_N)$

$p(t)$ unconstrained

In QFT, promote q^i to field amplitudes $\phi(\vec{x})$,

$$\langle \phi_b(\vec{x}) | e^{-iHT} | \phi_a(\vec{x}) \rangle = \int D\phi\, D\pi \exp\left(i \int_0^T d^4x \left(\pi\dot{\phi} - \tfrac{1}{2}\pi^2 - \tfrac{1}{2}(\nabla\phi)^2 - V(\phi)\right)\right)$$

with $\phi(\vec{x}, 0) = \phi_a(\vec{x})$, $\phi(\vec{x}, T) = \phi_b(\vec{x})$.

Complete square + evaluate $D\pi$ integral.

$$\langle \phi_b(\vec{x}) | e^{-iHT} | \phi_a(\vec{x}) \rangle = \int D\phi \exp\left[i \int_0^T d^4x\, \mathcal{L}\right]$$

$$\mathcal{L} = \tfrac{1}{2} (\partial_\mu \phi)^2 - V(\phi)$$

Ansatz:

$$\langle \Omega | T \phi(x_1) \phi(x_2) | \Omega \rangle$$

Peskin + Schroeder Sect. 9.2

$$= \lim_{T \to \infty(1-i\epsilon)} \frac{\int D\phi\, \phi(x_1) \phi(x_2) \exp\left[i \int_{-T}^{T} d^4x\, \mathcal{L}\right]}{\int D\phi \exp\left[i \int_{-T}^{T} d^4x\, \mathcal{L}\right]}$$

Derived from

$$\int D\phi(x) = \int D\phi_1(\vec{x}) \int D\phi_2(\vec{x}) \int_{\substack{\phi(x_1^0, \vec{x}) = \phi_1(\vec{x}) \\ \phi(x_2^0, \vec{x}) = \phi_2(\vec{x})}} D\phi(x)$$

Time-slice

Generalizations to higher orders: add more factors.

Instead of adding more ϕ, do functional derivatives with source term that defines generating functional.

Define $\dfrac{\delta J(y)}{\delta J(x)} = \delta^{(4)}(x-y)$

$$\frac{\delta}{\delta J(x)} \int d^4y\, J(y) \phi(y) = \phi(x)$$

$$\frac{\delta}{\delta J(x)} \exp\left[i\int d^4y\, J(y)\phi(y)\right] = i\phi(x) \exp\left[i\int d^4y\, J(y)\phi(y)\right]$$

$$\frac{\delta}{\delta J(x)} \int d^4y\, \partial_\mu J(y)\, V^\mu(y) = -\partial_\mu V^\mu(x)$$

by IbP (integration by parts)

Define generating functional

$$Z[J] \equiv \int \mathcal{D}\phi\, \exp\left[i\int d^4x\, (\mathcal{L} + J(x)\phi(x))\right]$$

Then, n-pt. correlation fcns. are

$$\langle \Omega | T\phi(x_1)\cdots\phi(x_n) | \Omega \rangle = \frac{1}{Z_0} \left(\frac{-i\delta}{\delta J(x_1)}\right) \cdots \left(\frac{-i\delta}{\delta J(x_n)}\right) Z[J] \Big|_{J=0}$$

$$Z_0 = Z[J=0]$$

Each insertion of $\frac{-i\delta}{\delta J(x_i)}$ brings down a factor of $\phi(x_i)$ to extract correlation fcn.