Lecture 11.

QFT II.

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(p.1)

Last time: Review of gauge symmetry & began discussion of non-Abelian gauge symmetry

Today: Group theory & representation theory

Motivation: From $\Psi(x) \to V(x)\Psi(x)$, with $V(x) \sim 1 + i\alpha^a(x) t^a + \mathcal{O}(\alpha^2)$

this describes continuous group, "Lie group"

Basic group theory

Groups are sets of elements with a multiplication rule $\cdot$.

1. If $x, y \in G$, then $x \cdot y \in G$.
2. There is an identity $e \cdot x = x \cdot e = x \quad \forall x$.
3. $\forall x \in G$, inverse exists & $x \cdot x^{-1} = x^{-1} \cdot x = e$
4. Associativity, $a \cdot (b \cdot c) = (a \cdot b) \cdot c$.

Distinguish Abelian vs. non-Abelian groups.

Abelian: Group operation is commutative for all group elements.

Non-abelian: Group operation is non-commutative.

Representations: Can map set elements such that algebraic mapping preserves group multiplication table.

$$D(x)\, D(y) = D(x \cdot y)$$

In physics, will use unitary operator on Hilbert space as D. For example: $D(n) = e^{in\theta}$ for group $\mathbb{Z}$ under addition

Introduce similarity transformation:

$$D_2(x) = S D_1(x) S^{-1} \quad \forall x \in G.$$

Rep. is reducible if equivalent (can find a sim. transformation) s.t. D_2 is in block diagonal form.

Then $D_2 = D_A \oplus D_B$ with vector space D_2 breaking into orthogonal subspaces. Otherwise, rep. is irreducible.

If we require group elements labeled by continuous parameters s.t. any infinitesimal group element g follows $g(\alpha) = 1 + i\alpha^a t^a + O(\alpha^2)$

α^a: infinitesimal group parameters

t^a: generators, Hermitian operators

This type of continuous group is a Lie group.

Generators span space of ~~inite~~ infinitesimal group transformations, so provides a basis to define:

$$[t^a, t^b] = i f^{abc} t^c$$

f^{abc}: structure constants.

Lie Algebra: vector space spanned by t^a ~~+ oppe~~ operation of commutation.

Jacobi identity:

$$[t^a, [t^b, t^c]] + [t^b, [t^c, t^a]] + [t^c, [t^a, t^b]] = 0$$

$$\Rightarrow f^{ade} f^{bcd} + f^{bde} f^{cad} + f^{cde} f^{abd} = 0$$

Global properties of groups can be different even with same Lie algebra.

Focus on compact Lie groups, since transform finite set of fields.

Given a general Lie algebra, can separate out all commuting generators (center of group), each generates $U(1)$.

If group has no $U(1)$ factors, algebra is semi-simple.

~~No~~ non-trivial invariant subalgebra: algebra is simple.

Complete classification of all compact simple Lie algebras (Cartan + Killing)

1) $U(N) = SU(N) \times U(1)$

$SU(N)$ = unitary $N \times N$ transformations, $\det U = 1$ $\quad \mathrm{tr}[t^a] = 0$, $N^2 - 1$ matrices.

2.) $O(N) = SO(N) \times$ reflection

$SO(N)$: unitary $N \times N$ transformations, preserve symmetric inner product.

rotation group in N dims.

$\frac{N(N-1)}{2}$ generators.

3) $Sp(N)$: unitary $N \times N$ that preserve $\eta_a \epsilon_{ab} \xi_b$, $\epsilon_{ab} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

N even.

$\frac{N}{2} \times \frac{N}{2}$ blocks

$\frac{N(N+1)}{2}$ generators

4.) Exceptional: G_2, F_4, E_6, E_7, E_8.

Given rep. r, $\phi \to (1 + i\alpha^a t_r^a)\phi$

$\exists$ conj. rep. $\bar{r}$ $\phi^* \to (1 - i\alpha^a (t_r^a)^*)\phi^*$

$t_{\bar{r}}^a = -(t_r^a)^* = -(t_r^a)^T$ since t^a are Hermitian.

If $\exists U$, s.t. $t_{\bar{r}}^a = U t_r^a U^\dagger$, then r is a real representation.

Then if η, ξ are in same rep. r, $G_{ab} \eta_a \xi_b$ is invariant.

For $G_{ab} = G_{ba}$, then r is strictly real. For $G_{ab} = -G_{ab}$, then r is pseudoreal.

Ex. $SU(2)$: $v_a w_a$ is real

$\epsilon^{ab} \eta_a \xi_b$ is pseudoreal.

Dimension of rep. = dim. of vector space on which it acts.

Representation matrices change depending on rep.: t_r^a.

Define $D^{ab} = tr[t_r^a t_r^b] = C(r) \delta^{ab}$

Also, $f^{abc} = \frac{-i}{C(r)} tr\{[t_r^a, t_r^b], t_r^c\}$

f^{abc} is totally anti-symmetric

Main reps.:

fundamental N or $\square$, N-dim complex vector

anti-fundamental $\bar{N}$ or $\bar{\square}$

adjoint

For $SU(2)$, $\square$ & $\bar{\square}$ not distinct.

Adjoint: Note f^{abc} provide representation matrices for adjoint $(t^b_G)_{ac} = i f^{abc}$

Adjoint is always real. $t^a_G = -(t^a_G)^*$

$$d(G) = \begin{cases} N^2-1 & SU(N) \\ N(N-1)/2 & SO(N) \\ N(N+1)/2 & Sp(N) \end{cases}$$

For any simple Lie algebra, $T^2 = T^a T^a$.

(analogous to total spin J^2 in $SU(2)$)

Commutes with all T^b.

$$t^a_r t^a_r = C_2(r)\, \mathbb{1}$$

↳ $d(r) \times d(r)$ unit matrix

$C_2(r)$ = quadratic Casimir

$$d(r)\, C_2(r) = d(G)\, C(r)$$

$C(N) = \frac{1}{2}$ $\quad C_2(N) = \frac{N^2-1}{2N}$

$C(G) = N$ $\quad C_2(G) = N$.