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Theoretische Elementarteilchenphysik

Quantum Field Theory II.

Lecture 1.

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Welcome + Announcements

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Exams: Oral, 50% of HW points required to register.

reader.uni-mainz.de for HW, syllabus, solutions, notes.

Introduction + Motivation to QFT II.

Physics – central goal is making quantitative predictions about natural phenomena.

Earliest example: parabolic motion in classical mechanics.

Eqn. from Newton's laws allowed prediction of speed + position of thrown ball.

Quantum physics: Pushing to apply classical knowledge to new regime of atomic systems, for example, led to failure of ability to measure $\hat{x}$ and $\hat{p}$ simultaneously

$\Rightarrow$ Heisenberg uncertainty principle $[\hat{x}, \hat{p}] = i\hbar \neq 0$

Led to birth of quantum operators tied to linear algebra $\Rightarrow$ behavior of quantum operators on wavefcns exactly same as linear algebra systems. Eigenvalues become quantum numbers when operator commutes w/ Hamiltonian.

In QFT, the analogous situation occurs when moving from free quantum field theory to interacting field theory. In free QFT, we can construct Green's fcns. of the appropriate Klein-Gordon and Dirac operators to solve the Hamiltonian and build arbitrary, complicated systems. The free Hamiltonian then determines the time evolution. This was explored in QFT I from the path integral approach, for example.

When we add interactions to free theory, we can typically / generically ~~cannot~~ no longer solve the Hamiltonian. As a consequence, we appeal to a procedure of perturbative field theory. Even accepting the approximations that the ground states of the free theory and the interacting theory are equivalent, we typically resum / approximate the effects of the interaction as a perturbative expansion:

Consider QED:

$$\mathcal{L} = \frac{-1}{4} F_{\mu\nu} F^{\mu\nu} + \overline{\Psi}(i\not{D} - m)\Psi$$

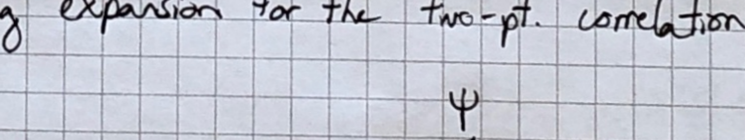
$\not{D} = \gamma^\mu(\partial_\mu + iQeA_\mu)$ covariant derivative

$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$

Q = electric charge of Ψ

e = gauge coupling $\left(\alpha = \frac{e^2}{4\pi} \right.$ ~~hyperfine~~ fine structure const.$)$

The non-trivial interaction between Ψ and A_μ gives the following expansion for the two-pt. correlation fcn. of photons:

$\Pi^{\mu\nu}(q^2) =$ 

Recall $\Pi^{\mu\nu}_{\substack{1\text{-loop}\\ 1PI}}$ is the bubble of Ψ with truncated photons.

Analyzing the 1-loop diagram, we get the integral over virtual momentum:

$$\Pi^{\mu\nu}_{1PI} \sim (-ie)^2 \int \frac{d^4k}{(2\pi)^4} \mathrm{Tr}\left[\left[\frac{i(\not{k}+m)}{k^2-m^2}\right]\gamma_\mu\left[\frac{i(\not{k}+\not{p}+m)}{(k+p)^2-m^2}\right]\gamma_\nu\right] \times (-1)$$

The main issue is that integral is quadratically divergent for $k \to \infty$!

Such UV divergent integrals can be rendered finite by extracting the essential part of the UV divergence (via dimensional regularization, Pauli-Villars regularization, or a different regularization prescription), and then interpreting finite part as a physical property of the amplitude (i.e. something that can be measured experimentally.)

Caution: Mathematically, this is very dangerous. In Pauli-Villars, for example, you are essentially subtracting two infinities to get a finite answer. But you should in mind that this requires a principle valve prescription without which there is no control over the value of the finite answer.)

The key point about the infinities / UV divergences is that different loop calculations in a given theory can <u>match</u> the infinities in such a way that all of the remaining finite parts are physical properties/ observable. As a specific case, in QED, the UV divergences in the photon 2pt. correlation fcn, the electron 2 pt. correlation fcn., and the $e^-e^+\gamma$ vertex are all matched to WFR constants $Z_{1,2}$ and a vertex counterterm Z_3.

[The electron mass is also renormalized.]

Because the UV divergences are "universal" for observers at any energy scale (i.e. experiments running at different energies), the finite answers measured by different observers are hence related by <u>renormalization flow.</u> Establishing the renormalization group equation (Callen-Symanzik equation) is the first main result of this course.

We begin by summarizing the main results of fields in QFT I to fix our conventions. (Ref. Peskin + Schroeder)

Scalar fields (solns. to Klein-Gordon)

[real] $$\phi(\vec{x}) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E}} \left(a_p + a^\dagger_{-p}\right) e^{-ip\cdot x}$$

$$\pi(\vec{x}) = \int \frac{d^3p}{(2\pi)^3} (-i)\sqrt{\frac{E}{2}} \left(a_{\vec{p}} - a^\dagger_{-\vec{p}}\right) e^{-ip\cdot x}$$

with $[a_{\vec{p}}, a^\dagger_{\vec{p}'}] = (2\pi)^3 \delta^{(3)}(\vec{p}-\vec{p}')$

$$\left.\begin{aligned} [\phi(\vec{x}), \pi(\vec{y})] &= i\,\delta^{(3)}(\vec{x}-\vec{y}) \\ [\phi(\vec{x}), \phi(\vec{y})] &= [\pi(\vec{x}), \pi(\vec{y})] = 0 \end{aligned}\right\} \text{quantization condition for } \phi + \pi$$

$a_{\vec{p}}$: annihilation of momentum eigenstate $\vec{p}$

$a^\dagger_{\vec{p}}$: creation " " " "

$\phi(\vec{x})$ creates particle at position $\vec{x}$.

Propagator: $$D(x-y) = \langle 0 | T\phi(x)\phi(y) | 0 \rangle = \int \frac{d^4p}{(2\pi)^4} \frac{i}{p^2 - m^2 + i\epsilon} e^{-ip\cdot(x-y)}$$

D is Green's fcn. of Klein-Gordon operator

$$(\partial^2 + m^2)\,\phi = -i\,\delta^{(4)}(x-y)$$

Fermion fields: (Dirac eqn.)

$$\psi(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec{p}}}} \sum_s \left(a^s_{\vec{p}}\, u^s(p)\, e^{-ip\cdot x} + b^{s\dagger}_{\vec{p}}\, v^s(p) e^{ip\cdot x} \right)$$

$$\bar{\psi}(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec{p}}}} \sum_s \left(b^s_{\vec{p}}\, \bar{v}^s(p) e^{-ip\cdot x} + a^s_{\vec{p}}\, \bar{u}^s(p)\, e^{ip\cdot x} \right)$$

$$\{ a^r_{\vec{p}}, a^{s\dagger}_{\vec{q}} \} = \{ b^r_{\vec{p}}, b^{s\dagger}_{\vec{q}} \} = (2\pi)^3 \delta^{(3)}(\vec{p}-\vec{q})\, \delta^{rs}$$

$$\{ \psi_a(\vec{x}), \psi^\dagger_b(\vec{y}) \} = \delta^{(3)}(\vec{x}-\vec{y})\, \delta_{ab}$$

$$\{ \psi_a(\vec{x}), \psi_b(\vec{y}) \} = \{ \psi^\dagger_a(\vec{x}), \psi^\dagger_b(\vec{y}) \} = 0$$

$a^{s\dagger}_{\vec{p}}$ creates fermions, $b^{s\dagger}_{\vec{p}}$ creates antifermions.

$$s=1,2: \quad u^s(p) = \begin{pmatrix} \sqrt{p\cdot\sigma}\, \xi^s \\ \sqrt{p\cdot\bar{\sigma}}\, \xi^s \end{pmatrix} \qquad v^s(p) = \begin{pmatrix} \sqrt{p\cdot\sigma}\, \eta^s \\ -\sqrt{p\cdot\bar{\sigma}}\, \eta^s \end{pmatrix}$$

$$S(x-y) = \int \frac{d^4p}{(2\pi)^4} \frac{i(\not{p}+m)}{p^2-m^2+i\epsilon} e^{-ip\cdot(x-y)}$$

$$= \langle 0 | T\, \psi(x)\, \bar{\psi}(y) | 0 \rangle$$

Again, $S(x-y)$ is Green's fn. for Dirac operator.

Cross sections

$$d\sigma = \frac{1}{2E_A\, 2E_B} \frac{1}{|v_A - v_B|} \left(\prod_f \frac{d^3p_f}{(2\pi)^3} \frac{1}{2E_f} \right) |\mathcal{M}(p_A, p_B) \to \{p_f\}|^2 \cdot (2\pi)^4 \delta^{(4)}\Big(p_A + p_B - \sum_f p_f\Big)$$

Width

$$d\Gamma = \frac{1}{2m_A} \left(\prod_f \frac{d^3p_f}{(2\pi)^3} \frac{1}{2E_f} \right) |\mathcal{M}(m_A \to \{p_f\})|^2 \cdot (2\pi)^4 \delta^{(4)}\Big(p_A - \sum_f p_f\Big)$$

S-matrix. Takes in states to out states in Hilbert ~~states~~ space.

Necessarily unitary. $S = \mathbb{1} + iT$

$$\langle \vec{p}_1 \vec{p}_2 \cdots | iT | \vec{k}_A \vec{k}_B \rangle = (2\pi)^4 \delta^{(4)}\Big(k_A + k_B - \sum p_f\Big)\, i\mathcal{M}(k_A, k_B \to \{p_f\})$$

In calculations, Feynman diagrams represent $\mathcal{M}$.

Requires interactions

Mass terms, e.g. $\frac{m^2}{2}\phi^2$, $m\bar{\Psi}\Psi$, are simplistic interactions

In $D=4$, three or more fields give non-trivial interactions.

Example: $g\,\bar{e}\gamma_\mu A^\mu e$ from QED.

$y\,H\,\bar{L}_L e_R$ + h.c. from Standard Model

$\frac{\lambda}{4!}\phi^4$ from scalar field theory.

Begin renormalization.

Critical issue: In an interacting theory, how do you relate the eigenstates of the interacting Hamiltonian with eigenstates of free Hamiltonian?

$|0\rangle$: ground state of free

$|\Omega\rangle$: ground state of interacting

Recall $\langle 0|T\phi(x)\phi(y)|0\rangle$: 2 pt. correlation fcn. in free theory. (propagator) - amplitude for particle to go from y to x.

$\langle \Omega|T\phi(x)\phi(y)|\Omega\rangle$: 2 pt. correlation fcn. in interacting theory.

Schwartz p.70-74

Generalize to n-particles: defined for asymptotic free states.

$\langle f|\mathcal{M}|i\rangle$ is S-matrix element with δ-fcn. taken out.

$$\underset{t=+\infty}{\langle f|}\, S-\mathbb{1}\, \underset{t=-\infty}{|i\rangle} = i(2\pi)^4\delta^{(4)}(\Sigma p)\,\mathcal{M}$$

$$|i\rangle = \sqrt{2E_1}\sqrt{2E_2}\; a^\dagger_{p_1}(-\infty)\, a^\dagger_{p_2}(-\infty)\,|\Omega\rangle$$

$$|f\rangle = \sqrt{2E_3}\ldots\sqrt{2E_n}\; a^\dagger_{p_3}(+\infty)\, a^\dagger_{p_n}(+\infty)\,|\Omega\rangle$$

Assume $|f\rangle \neq |i\rangle$, so discard $\mathbb{1}$.

$$\langle f|S|i\rangle = 2^{n/2}\sqrt{E_1\ldots E_n}\;\langle\Omega|\, a_{p_3}(\infty)\ldots a_{p_n}(\infty)\, a^\dagger_{p_1}(-\infty)\, a^\dagger_{p_2}(-\infty)\,|\Omega\rangle$$

To progress, need to prove

$$i\int d^4x\; e^{ip\cdot x}\,(\Box + m^2)\,\phi(x) = \sqrt{2E_p}\,\big(a_p(\infty) - a_p(-\infty)\big)$$

for asymptotic free states. Assume fields die off at $|\vec{x}| = \pm\infty$, so can integrate by parts in $\vec{x}$.

$$i\int d^4x\, e^{ip\cdot x}\,(\Box+m^2)\,\phi(x) \equiv i\int d^4x\, e^{ip\cdot x}\left(\partial_t^2-\vec{\partial}_x^{\,2}+m^2\right)\phi(x)$$

IBP, require $\left(\partial_x\phi\, e^{-ip\cdot x}\right)\Big|_{-\infty}^{\infty}=0$ $\quad = i\int d^4x\, e^{ip\cdot x}\left(\partial_t^2+\vec{p}^{\,2}+m^2\right)\phi(x)$

$$= i\int d^4x\, e^{ip\cdot x}\left(\partial_t^2+E_p^2\right)\phi(x)$$

Now use $\partial_t\left[e^{ip\cdot x}(i\partial_t+E_p)\,\phi(x)\right] = \left[iE_p e^{ip\cdot x}(i\partial_t+E_p)+e^{ip\cdot x}(i\partial_t^2+E_p\partial_t)\right]\phi$

$$= i e^{ip\cdot x}\left(\partial_t^2+E_p^2\right)\phi(x)$$

independent of B.Cs.

Together, $i\int d^4x\, e^{ip\cdot x}(\Box+m^2)\phi(x) = \int d^4x\, \partial_t\left[e^{ip\cdot x}(i\partial_t+E_p)\phi(x)\right]$

$$= \int dt\, \partial_t\left[e^{iE_p t}\int d^3x\, e^{-i\vec{p}\cdot\vec{x}}(i\partial_t+E_p)\phi(x)\right]$$

Integrand is total derivative, evaluate at bdry $t=\pm\infty$.

Note $a_p(t)$ + $a_p^\dagger(t)$ operators are time-independent at $t\to\pm\infty$

Calculate

$$\int d^3x\, e^{-i\vec{p}\cdot\vec{x}}(i\partial_t+E_p)\,\phi(x)$$

$$= \int d^3x\, e^{-i\vec{p}\cdot\vec{x}}(i\partial_t+E_p)\int\frac{d^3k}{(2\pi)^3}\frac{1}{\sqrt{2E_k}}\left(a_k(t)e^{-ik\cdot x}+a_k^\dagger(t)e^{ik\cdot x}\right)$$

$$\star = \int\frac{d^3k}{(2\pi)^3}\int d^3x\left[\frac{(E_k+E_p)}{\sqrt{2E_k}}a_k(t)e^{-ik\cdot x}e^{-i\vec{p}\cdot\vec{x}} + \frac{(-E_k+E_p)}{\sqrt{2E_k}}a_k^\dagger(t)e^{ik\cdot x}e^{-i\vec{p}\cdot\vec{x}}\right]$$

using $\partial_t a_k(t)=0$ for bdrys.

Now, $\int d^3x\, e^{-ik\cdot x}e^{-i\vec{p}\cdot\vec{x}} = e^{-iE_k t}\,\delta^{(3)}(\vec{p}-\vec{k})\,(2\pi)^3$

and similar for second term:

$$\int d^3x\, e^{+ik\cdot x}e^{-i\vec{p}\cdot\vec{x}} = e^{iE_k t}\,\delta^{(3)}(\vec{p}+\vec{k})\,(2\pi)^3$$

We get $\star = \sqrt{2E_p}\, a_p(t)\, e^{-iE_p t}$

So, $i\int d^4x\, e^{ip\cdot x}(\Box+m^2)\phi(x) = \int dt\,\partial_t[\ldots]$

$$= \sqrt{2E_p}\left(a_p(\infty)-a_p(-\infty)\right)$$

Also, by h.c.

$$-i\int d^4x\, e^{-ip\cdot x}(\Box+m^2)\,\underset{\uparrow}{\phi}(x) = \sqrt{2E_p}\left(a_p^\dagger(\infty)-a_p^\dagger(-\infty)\right)$$

real field

So, $\langle f|S|i\rangle = \sqrt{2^n}\sqrt{E_1\ldots E_n}\ \langle\Omega|a_{p_3}(\infty)\ldots a_{p_n}(\infty)\,a^\dagger_{p_1}(-\infty)\,a^\dagger_{p_2}(-\infty)|\Omega\rangle$

$$= \sqrt{2^n}\sqrt{E_1\ldots E_n}\ \langle\Omega|\,T\{a_{p_3}\ldots a_{p_n}a^\dagger_{p_1}a^\dagger_{p_2}\}|\Omega\rangle$$

$$= \sqrt{2^n}\sqrt{E_1\ldots E_n}\ \langle\Omega|\,T\{[a_{p_3}(\infty)-a_{p_3}(-\infty)]\ldots[a_{p_n}(\infty)-a_{p_n}(-\infty)]$$
$$[a^\dagger_{p_1}(-\infty)-a^\dagger_{p_1}(+\infty)]\,[a^\dagger_{p_2}(-\infty)-a^\dagger_{p_2}(\infty)]\}|\Omega\rangle$$

All new a_i will shuffle to annihilate on vacuum.

$$\langle f|S|i\rangle = \langle p_3\ldots p_n|S|p_1p_2\rangle$$
$$= \left[i\int d^4x_1\,e^{-ip_1\cdot x_1}(\Box_1+m^2)\right]\ldots\left[i\int d^4x_n\,e^{ip_n\cdot x_n}(\Box_n+m^2)\right]$$
$$\langle\Omega|\,T\{\phi(x_1)\ldots\phi(x_n)\}|\Omega\rangle$$

Lehmann - Symanzik - Zimmermann reduction formula.

Discussion + interpretation

For S-matrix, multiply correlation fcn. by $\Box+m^2$ + Fourier transform.

For $\phi(x)$ as free fields, $(\Box+m^2)\phi=0$

But one-particle states have propagators $\frac{1}{\Box+m^2}\Rightarrow\frac{1}{p^2-m^2}$ → will then extract residues at pole, will differ b/t conventions (on-shell, subtraction)

Remark: unitary theories require simple poles, so correlation fcn. can only fall off as p^{-2} for large p^2.

Note: for fields with spin, must also multiply by spinor $[u^s(p)]$ or polarization vector $[\epsilon^\mu(k)]$ to project desired spin state.