Functional Method + Deriving Feynman Rules Primer
Quantum Field Theory
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In this primer, we explain how to derive Feynman rules from the functional method.

The starting point is

$$\langle \Omega | T \phi_H(x_1) \phi_H(x_2) \dots \phi_H(x_n) | \Omega \rangle$$

$$= \lim_{T \to \infty(1-i\epsilon)} \frac{\int \mathcal{D}\phi \, \phi(x_1) \phi(x_2) \dots \phi(x_n) \exp\left[i \int_{-T}^{T} d^4x \, \mathcal{L} \right]}{\int \mathcal{D}\phi \exp\left[i \int_{-T}^{T} d^4x \, \mathcal{L} \right]}$$

In this expression, we interpret the denominator as simply the path integral expression for the vacuum to vacuum transition. This can be justified by considering the numerator without any $\phi(x_i)$ fields; the RHS is then just 1 and the LHS is the $\langle \Omega | \Omega \rangle$ vacuum-to-vacuum correlation fcn.

Following the discussion in P+S, we can use a generating functional to simplify + streamline the approach.
Define $\quad Z[J] = \int \mathcal{D}\phi \exp\left[i \int d^4x \left(\mathcal{L} + J(x) \phi(x) \right) \right]$

where $J(x)\phi(x)$ is a source term for ϕ. Note that the earlier denominator is $Z[J=0] \equiv Z_0$.

We can explicitly solve $Z[J]$ where $\mathcal{L}$ is the free Lagrangian:

$$Z[J] = Z_0 \exp\left[-\frac{1}{2} \int d^4x \, d^4y \, J(x) D_F(x-y) J(y) \right]$$

The generating functional of free theory is a scalar propagator connecting two sources.

With the language of the generating functional, n-pt. correlation fcns. are functional derivatives

$$\langle 0 | T\phi(x_1)\phi(x_2)\dots\phi(x_n)|0\rangle = \frac{1}{Z_0}\left(\frac{-i\delta}{\delta J(x_1)}\frac{-i\delta}{\delta J(x_2)}\dots\frac{-i\delta}{\delta J(x_n)}\right) Z[J]\Big|_{J=0}$$

Note 1: We only have one source term since we have one degree of freedom. If we had a complex field, we would have to add two (conjugate) source terms,

$$\int D\phi^* D\phi \exp\left\{ i\int d^4x\, d^4y \left[\phi^*(x) M(x,y)\phi(y)\right] + i\int d^4x \left[J^*(x)\phi(x) + \phi^*(x)J(x)\right]\right\}$$
$$= N \frac{1}{\det M} \exp\left\{ -i\int d^4x\, d^4y\, J^*(x) M^{-1}(x,y) J(y)\right\}$$

See problem 14.1 of Schwartz.

We will also see that fermions also require separate source terms for particles & antiparticles.

Note 2: These correlation functions are all calculated in position space. To convert to momentum space, we need to Ⓐ use the Lehmann-Symanzik-Zimmermann reduction formula to relate the correlation fcn. to the S-matrix and to select out the on-shell states as asymptotic states and Ⓑ Fourier transform to momentum space.

The simplest demonstration of this is studying the two-pt. correlation fcn. for free scalar theory.

Recall that functional derivatives behave

$$\frac{\delta J(y)}{\delta J(x)} = \delta^{(4)}(x-y)$$

so $\frac{\delta}{\delta J(x)} \int d^4y \, J(y)\phi(y) = \int d^4y \, \delta^{(4)}(x-y)\phi(y) = \phi(x)$

and $\frac{\delta}{\delta J(x)} \exp\left[i\int d^4y \, J(y)\phi(y)\right] = i\phi(x)\exp\left[i\int d^4y \, J(y)\phi(y)\right]$

and $\frac{\delta}{\delta J(x)} \int d^4y \, \partial_\mu J(y) \, V^\mu(y) = -\partial_\mu V^\mu(y)$

Then, we have

$$\langle 0 | T\phi(x_1)\phi(x_2) | 0 \rangle$$

$$= \frac{1}{Z_0}\left(\frac{-i\delta}{\delta J(x_1)}\frac{-i\delta}{\delta J(x_2)}\right) Z_0 \exp\left[-\frac{1}{2}\int d^4x\, d^4y \, J(x) D_F(x-y) J(y)\right]\Bigg|_{J=0}$$

$$= \frac{1}{Z_0}\left(\frac{-\delta}{\delta J(x_1)}\right)\left(-\frac{1}{2}\int d^4y \, D_F(x_2-y) J(y) \, Z[J] - \frac{1}{2}\int d^4x \, J(x) \, D_F(x-x_2) \, Z[J]\right)\Bigg|_{J=0}$$

$$= \frac{-1}{Z_0}\Bigg(-\frac{1}{2} D_F(x_2-x_1) \, Z[J]$$

$$-\frac{1}{2}\left(\int d^4y \, D_F(x_2-y) J(y)\right)\cdot\left(-\frac{1}{2}\int d^4z \, J(z) D_F(z-x_1) Z[J] - \frac{1}{2}\int d^4z \, D_F(x_1-z) J(z) Z[J]\right)$$

$$-\frac{1}{2} D_F(x_1-x_2) \, Z[J]$$

$$-\frac{1}{2}\left(\int d^4x \, J(x) D_F(x-x_2)\right)\left(-\frac{1}{2}\int d^4z \, J(z) D_F(z-x_1) Z[J] - \frac{1}{2}\int d^4z \, D_F(x_1-z) J(z) Z[J]\right)\Bigg)\Bigg|_{J=0}$$

Setting $J=0$, we get $Z[J] = Z_0$ and all extra terms vanish,

$$= \frac{1}{2} D_F(x_2-x_1) + \frac{1}{2} D_F(x_1-x_2)$$

Note $D_F(x_2-x_1) = \begin{cases} D(x_2-x_1) & \text{for } x_2^0 > x_1^0 \\ D(x_1-x_2) & \text{for } x_2^0 < x_1^0 \end{cases}$ $\Rightarrow D_F(x_1-x_2) = D_F(x_2-x_1)$

$$D_F(x_1-x_2) = \begin{cases} D(x_1-x_2) & \text{for } x_1^0 > x_2^0 \\ D(x_2-x_1) & \text{for } x_1^0 < x_2^0 \end{cases}$$

So we get

$$\langle 0| T\phi(x_1)\,\phi(x_2)|0\rangle = D_F(x_1-x_2)$$

$$= \int \frac{d^4p}{(2\pi)^4} \frac{i}{p^2-m^2+i\epsilon} e^{-ip\cdot(x_1-x_2)}$$

This is the fourier transform of the regular propagator

x_1 - - - - - x_2 $\qquad \dfrac{i}{p^2-m^2+i\epsilon}$

We now add an interaction: $\mathcal{L}_{int} = -\frac{\lambda}{4!}\phi^4$

We have

$$\langle \Omega| T\phi(x_1)\,\phi(x_2)|\Omega\rangle$$

$$= \frac{1}{Z_0}\left(\frac{-i\delta}{\delta J(x_1)}\frac{-i\delta}{\delta J(x_2)}\right) Z[J]\Big|_{J=0}$$

where $Z[J] = \int D\phi \exp\left[i\int d^4x\left(\mathcal{L}_{free} + J\phi + \mathcal{L}_{int}\right)\right]$

$$= \int D\phi \exp\left[i\int d^4x\left(\mathcal{L}_{free} + J\phi\right)\right] \exp\left[i\int d^4x\,\mathcal{L}_{int}\right]$$

$$\approx \int D\phi \exp\left[i\int d^4x\left(\mathcal{L}_{free} + J\phi\right)\right]$$
$$\left(1 + i\int d^4x\,\mathcal{L}_{int} + \frac{\left(i\int d^4x\,\mathcal{L}_{int}\right)^2}{2} + \dots\right)$$

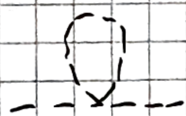
$$= Z_{free}[J]$$
$$+ i\int d^4x\left(-\frac{\lambda}{4!}\phi^4\right) Z_{free}[J] + \dots$$

Note that we can replace ϕ^4 in the $\mathcal{L}_{int}$ by the corresponding functional derivative of the free generating functional $Z_{free}[J] = Z_0 \exp\left[-\frac{1}{2}\int d^4x\,d^4y\, J(x) D_F(x-y) J(y)\right]$

We see $Z[J] \approx Z_{free}[J]$

$$+ i\int d^4x\left(-\frac{\lambda}{4!}\right)\left(\frac{-i\delta}{\delta J}\right)^4 Z_{free}[J] + \dots$$

Given that we focus on the 2-pt. correlation fcn, the $-\frac{\lambda}{4!}\phi^4$ interaction term shows up in the connected diagram

as . If we want the simple Feynman rule

of the 4 pt. interaction, we can consider the 4 pt. corr. fcn.

$$\langle\Omega| T\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)|\Omega\rangle = \frac{1}{Z_0}\left(\frac{-i\delta}{\delta J(x_1)}\,\frac{-i\delta}{\delta J(x_2)}\,\frac{-i\delta}{\delta J(x_3)}\,\frac{-i\delta}{\delta J(x_4)}\right)$$
$$\left[Z_{free}[J] + i\int d^4x\left(-\frac{\lambda}{4!}\right)\left(\frac{-i\delta}{\delta J}\right)^4 Z_{free}[J] + \dots\right]$$

As before, the LSZ formula extracts the one-particle states indicated by $D_F(x_1-x)\,D_F(x_2-x)\,D_F(x_3-x)\,D_F(x_4-x)$, which comes from the $\left(\frac{-i\delta}{\delta J(x_1)}\right)\cdots\left(\frac{-i\delta}{\delta J(x_4)}\right)$ functional derivatives.

Discard $Z_{free}[J]$ since it only gives propagator connectors. Focus on $i\int d^4x\left(\frac{-\lambda}{4!}\right)\phi^4\, Z_{free}[J]$ term.

$$\langle\Omega| T\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)|\Omega\rangle$$
$$\sim \frac{1}{Z_0}\left(\frac{-i\delta}{\delta J(x_1)}\right)\left(\frac{-i\delta}{\delta J(x_2)}\right)\left(\frac{-i\delta}{\delta J(x_3)}\right)\left(\frac{-i\delta}{\delta J(x_4)}\right)$$
$$+ i\int d^4x\left(\frac{-\lambda}{4!}\right)(\phi(x))^4\left[Z_0\exp\left[-\frac{1}{2}\int d^4a\, d^4b\, J(a) D_F(a-b) J(b)\right]\right]_{J=0}$$

$$= \frac{1}{Z_0}(+i)\int d^4x\left(\frac{-\lambda}{4!}\right)(\phi(x))^4\left(\frac{\delta}{\delta J(x_1)}\right)\left(\frac{\delta}{\delta J(x_2)}\right)\left(\frac{\delta}{\delta J(x_3)}\right)$$
$$\left(-\frac{1}{2}\int d^4b\, D_F(x_4-b) J(b)\, Z_{free}[J] - \frac{1}{2}\int d^4a\, J(a)\, D_F(a-x_4)\, Z_{free}[J]\right)\Big|_{J=0}$$

As before, each $\frac{\delta}{\delta J(x_i)}$ will bring down the corresponding $D_F(x_i - b) J(b)$ term or the $J(a) D_F(a - x_i)$ term. But remember that we also have $\phi^4(x) = \left(\frac{\delta}{\delta J(x)}\right)^4$ to perform too.

Given that we set $J=0$ at the end, then the only terms that remain are when $\frac{\delta}{\delta J(x_i)}$ give $D_F(x_i - b) J(b)$ or $J(a) D_F(a - x_i)$, and then we use $\frac{\delta}{\delta J(x)}$ to remove the remaining $J(b)$ or $J(a)$ and get $D_F(x_i - x) = D_F(x - x_i)$.

Explicitly, the terms are

$$\frac{1}{Z_0}(+i)\int d^4x \left(-\frac{\lambda}{4!}\right)\left(\frac{\delta}{\delta J(x)}\right)^4$$

combine $-\frac{1}{2}\int db\, D_F J(b)$ $-\frac{1}{2}\int da\, J(a) D_F$

$$\rightsquigarrow \left(-\int d^4b_4\, D_F(x_4 - b_4) J(b_4)\right)\left(-\int d^4b_3\, D_F(x_3 - b_3) J(b_3)\right)\left(-\int d^4b_2\, D_F(x_2 - b_2) J(b_2)\right)\left(-\int d^4b_1\, D_F(x_1 - b_1) J(b_1)\, Z_{free}[J]\right)\Bigg|_{J=0}$$

we have 4 cases for $\frac{\delta}{\delta J(x)}$ on $J(b_1) \ldots J(b_4)$

$$= \frac{1}{Z_0}(+i)\left(-\frac{\lambda}{4!}\right)\int d^4x\; 4!\, D_F(x_4 - x)\, D_F(x_3 - x)\, D_F(x_2 - x)\, D_F(x_1 - x)\, Z_{free}[J]\Big|_{J=0}$$

$$= -i\lambda \int d^4x\, D_F(x_4 - x)\, D_F(x_3 - x_4)\, D_F(x_2 - x)\, D_F(x_1 - x)$$

We recognize $\int d^4x\, D_F \ldots D_F$ as the position-space requirement for 4 propagators to meet at point x.

So, the momentum space Feynman rule is $-i\lambda$, corresponding to the Lagrangian term $\mathcal{L} = -\frac{\lambda}{4!}\phi^4$

Comment: the replacement of ϕ^4 by $\left(\frac{\delta}{\delta J(x)}\right)^4$ is the equivalent to performing a derivative of the interaction term by the external $\frac{\delta}{\delta J(x_i)}$ for each type of external field.

In position space, we get $D_F(x_i - x)$ propagators that meet at position x.