Lecture 5.

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QFT II.

Felix Yu

Last time: Refresher on QED renormalization
Ward identity and connection between vertex + electron field renorm. in QED.

Today: The renormalization group, continued. $\lambda\phi^4$ theory.
Callan-Symanzik equation

We continue our analysis of the renormalization group.
Consider the ϕ^4 Lagrangian:

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 - \frac{1}{2} m_0^2 \phi^2 - \frac{\lambda_0}{4!}\phi^4$$

We adopt wavefcn. rescaling $\phi = Z^{1/2}\phi_R$, where ϕ is the bare field, ϕ_R is the renormalized field.

$$\mathcal{L} = \frac{1}{2} Z (\partial_\mu \phi_R)^2 - \frac{1}{2} m_0^2 Z \phi_R^2 - \frac{\lambda_0}{4!} Z^2 \phi_R^4$$

Define $Z = 1 + \delta_Z$, where δ_Z is the counterterm.

$$\delta_m = m_0^2 Z - m_R^2$$

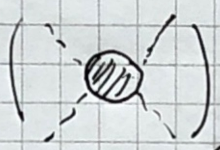
$$\delta_\lambda = \lambda_0 Z^2 - \lambda.$$

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi_R)^2 - \frac{1}{2} m_R^2 \phi_R^2 - \frac{\lambda}{4!}\phi_R^4$$
$$+ \frac{1}{2}\delta_Z (\partial_\mu \phi_R)^2 - \frac{1}{2}\delta_m \phi_R^2 - \frac{\delta_\lambda}{4!}\phi_R^4$$

We see that we have the original Lagrangian in the first line, where the second line is defined by extracting the divergences of the bare Lagrangian.

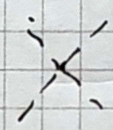
Renormalization conditions

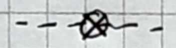
 $= \frac{i}{p^2-m^2} +$ non-singular at $p^2=m^2$

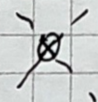
$\left(\right.$$\left.\right)_{\text{amputated}} = -i\lambda$ at $s=4m^2$, $t=u=0$

Feynman rules:

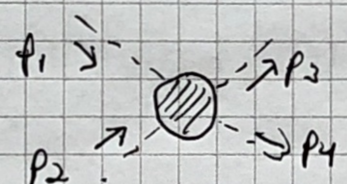
 $\frac{i}{p^2-m^2+i\epsilon}$

 $-i\lambda$

 $i(p^2\delta_Z-\delta_m)$

 $-i\delta_\lambda$

Calculate and impose $-i\lambda = \left(\right.$$\left.\right)_{\text{amp.}}$ at one-loop.



has many diagrams

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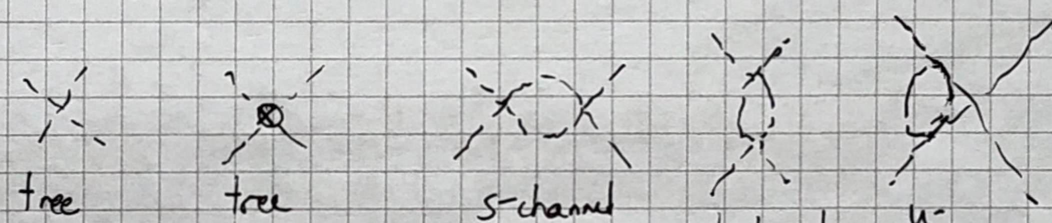
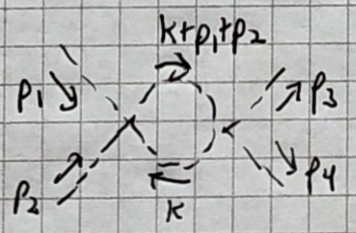


Calculate the s-channel diagram



$$i\mathcal{M}_s = \frac{(-i\lambda)^2}{2}\int \frac{d^4k}{(2\pi)^4}\,\frac{i}{k^2-m^2}\,\frac{i}{(k+p_1+p_2)^2-m^2}$$

symmetry factor

$$= \frac{(-i\lambda)^2}{2}(-1)\int_0^1 dx\, dy\;\delta(x+y-1)\int\frac{d^4k}{(2\pi)^4}\,\frac{1}{\left(k^2-m^2+x\,2k\cdot(p_1+p_2)+x\,(p_1+p_2)^2\right)^2}$$

$$= \frac{\lambda^2}{2}\int_0^1 dx\int\frac{d^dk}{(2\pi)^d}\,\frac{1}{\left(k^2+2k\cdot x(p_1+p_2)+x(p_1+p_2)^2-m^2\right)^2}$$

(3)

Shift

$l \equiv k + x(p_1+p_2)$

Complete the square to shift k to l:

We then define $\Delta_s = m^2 - x(p_1+p_2)^2 + x^2(p_1+p_2)^2$

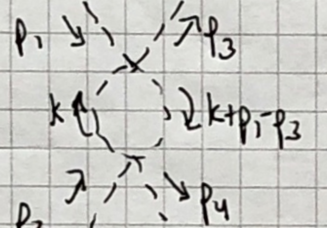
$$i\mathcal{M}_s = \frac{\lambda^2}{2}\int_0^1 dx \int \frac{d^d l}{(2\pi)^d}\frac{1}{(l^2-\Delta)^2}$$

for $d = 4-\epsilon$, we have $\displaystyle\int \frac{d^d l}{(2\pi)^d}\frac{1}{(l^2-\Delta)^n} = \frac{(-1)^n i}{(4\pi)^{d/2}}\frac{\Gamma(n-\frac{d}{2})}{\Gamma(n)}\left(\frac{1}{\Delta}\right)^{n-d/2}$

& use $\displaystyle\frac{\Gamma(2-d/2)}{(4\pi)^{d/2}}\left(\frac{1}{\Delta}\right)^{2-d/2} = \frac{1}{(4\pi)^2}\left(\frac{2}{\epsilon} - \log\Delta - \gamma + 4\pi + \mathcal{O}(\epsilon)\right)$

$$i\mathcal{M}_s = \frac{\lambda^2}{2}\frac{i}{16\pi^2}\int_0^1 dx\left(\frac{2}{\epsilon} - \log\Delta_s - \gamma + \log 4\pi + \mathcal{O}(\epsilon)\right)$$

Similarly,



gives $\Delta_t = x^2 t - xt + m^2$,

$t = (p_1-p_3)^2$

and the u-channel has $\Delta_u = x^2 u - xu + m^2$

$u = (p_1-p_4)^2$

Total

$$i\mathcal{M}_{tot} = -i\lambda + f(s) + f(t) + f(u) - i\delta_\lambda$$

Impose renormalization condition, so we need

$$i\delta_\lambda = f(s)\Big|_{s=4m^2} + f(t)\Big|_{t=0} + f(u)\Big|_{u=0} \quad \text{@ 1-loop order.}$$

Then,

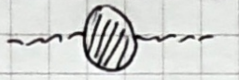
$$i\delta_\lambda = \frac{i\lambda^2}{32\pi^2}\left[\int_0^1 dx\left(\frac{6}{\epsilon} - 3\gamma + 3\log 4\pi - \log(m^2 - 4xm^2 + 4x^2m^2) - 2\log m^2\right)\right]$$

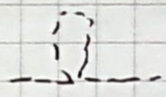
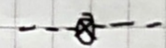
Plug back in:

$$i\mathcal{M}_{tot} = -i\lambda - \frac{i\lambda^3}{32\pi^2}\int_0^1 dx \log\left(\frac{m^2 - xs + x^2s^3}{m^2 - 4m^2x + 4m^2x^2}\right)$$
$$- \frac{i\lambda^3}{32\pi^2}\int_0^1 dx \log\left(\frac{m^2 - xt + x^2t^2}{m^2}\right) - \frac{i\lambda^2}{32\pi^2}\int_0^1 dx \log\left(\frac{m^2 - xu + x^2u^2}{m^2}\right)$$

We see that the divergences $\frac{2}{\epsilon}$ and finite contributions are cancelled.

Can check

 $= \dfrac{i}{p^2 - m^2 - M^2(p^2)}$

where we calculate $M^2(p^2)$ from  + 

and imposing $M^2(p^2)\Big|_{p^2=m^2} = 0$ & $\dfrac{d}{dp^2} M^2(p^2)\Big|_{p^2=m^2} = 0$

$$\Rightarrow \delta_Z = 0, \quad \delta_m = \frac{-\lambda}{2(4\pi)^{d/2}} \frac{\Gamma(1-\frac{d}{2})}{(m^2)^{1-d/2}}$$

Now, consider defining renormalization conditions differently.

For convenience, suppose we used a scale μ for s, t & u.

The relation between bare & renormalized Green's fns. is

$$\langle\Omega| T \phi_0(x_1) \dots \phi_0(x_n) |\Omega\rangle = Z^{n/2} \langle\Omega| T\phi(x_1) \dots \phi(x_n) |\Omega\rangle$$

Differentiate both sides wrt μ. The bare Green's fcn. has no μ-dependence, so LHS is zero.

$$0 = \mu \frac{d}{d\mu} G_n^{(0)} = \mu \frac{d}{d\mu}(Z^{n/2} G_n)$$

$$= \mu \frac{n}{2}\left(\frac{dZ}{d\mu}\right)\frac{Z^{n/2}}{Z} G_n + \mu Z^{n/2} \frac{\partial G_n}{\partial \mu}$$

$$+ \mu Z^{n/2} \left(\frac{\partial \lambda}{\partial \mu}\right) \frac{\partial G_n}{\partial \lambda}$$ for massless ϕ^4.

Recall chain rule

$$dG_n = \frac{\partial G}{\partial x} dx + \frac{\partial G}{\partial \lambda} d\lambda + \frac{\partial G}{\partial \mu} d\mu$$

$$\Rightarrow 0 = \left[n\left(\frac{1}{2} \frac{\mu}{Z} \frac{\partial Z}{\partial \mu}\right) + \mu \frac{\partial \lambda}{\partial \mu} \frac{\partial}{\partial \lambda} + \mu \frac{\partial}{\partial \mu} \right] G_n$$

This is the Callen-Symanzik equation

Interpretation: Describes the variation of the renormalized Green's fcn. wrt scale μ that defines renormalization conditions. We have to recover same theory if renorm. conditions at different scales "agree."

Two key coefficients in the differential equation:

$$\beta(\lambda) \equiv \mu \frac{\partial \lambda}{\partial \mu} = \frac{\partial \lambda}{\partial \log \mu} \qquad \text{"}\beta\text{- fcn."}$$

Gives the change in coupling strength as you vary scale.

$$\gamma(\lambda) \equiv \frac{1}{2} \frac{\mu}{Z} \frac{\partial Z}{\partial \mu} \qquad \text{anom. dim.}$$

Gives the deviation from classical scaling behavior of G_n.

For massive ϕ^4 theory, differential includes new terms:

$$0 = \left(\mu \frac{\partial}{\partial \mu} + \beta(\lambda) \frac{\partial}{\partial \lambda} + n\gamma(\lambda) + \gamma_{\phi^2} m^2 \frac{\partial}{\partial m^2} \right) G^{(n)}(\{p_i\}; \mu, \lambda, m^2)$$

where $\gamma_{\mathcal{O}} = \mu \frac{\partial}{\partial \mu} \log Z_{\mathcal{O}}(\mu)$ for operator $\mathcal{O}$ and

$$\mathcal{O}_{\text{bare}} = Z_{\mathcal{O}}(\mu)\, \mathcal{O}_{\text{ren.}}$$