Lecture 14.

QFT II.

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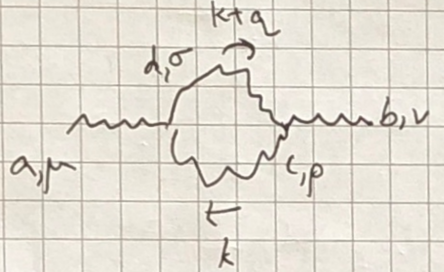
Last time: Calculate $f\bar{f} \to G^a G^b$ + $\Pi_{\mu\nu}(q^2)$ to demonstrate how ghosts are necessary in loop calculations in Yang-Mills theory to get gauge-invariant results.

Today: Finish $\Pi_{\mu\nu}(q^2)$ vacuum polarization diagram.



ignore extra Lorentz indices

$$\Pi_{\mu\nu}(q^2) = \frac{1}{2} g^2 \int \frac{d^4k}{(2\pi)^4} \frac{-i}{k^2} \frac{-i}{(k+q)^2} \left(f^{acd}\right)\left(g^{\mu\rho}(q-k)^\sigma + g^{\rho\sigma}(2k+q)^\mu + g^{\sigma\mu}(-k-2q)^\rho\right)$$

symmetry factor

$$\left(f^{bcd}\right)\left(\delta^\nu_\rho (k-q)_\sigma + g_{\rho\sigma}(-2k-q)_\nu + \delta^\nu_\sigma (k+2q)_\rho\right)$$

Use $f^{acd} f^{bcd} = C_2(G)\,\delta^{ab}$

Also, $$\frac{1}{k^2(k+q)^2} = \int dx \frac{1}{(k^2 + 2x k\cdot q + xq^2)^2}$$

Shift $\ell \equiv k + xq$

$$= \int dx \frac{1}{(\ell^2 + xq^2 - x^2 q^2)^2}$$

$$\Delta = x^2 q^2 - x q^2$$

$$\Pi_{\mu\nu} = -\frac{1}{2} g^2 C_2(G)\,\delta^{ab} \int \frac{d^d\ell}{(2\pi)^d} \frac{1}{(\ell^2-\Delta)^2}$$

$$\Big[-g^{\mu\nu}(q-k)^2 + (2k+q)^\mu (k-q)^\nu + (k-q)^\mu(-k-2q)^\nu$$
$$+ (q-k)^\mu(-2k-q)^\nu + (2k+q)^\mu(-2k-q)^\nu d + (-k-2q)^\mu(-2k-q)^\nu$$
$$+ (k+2q)^\mu (q-k)^\nu + (2k+q)^\mu (k+2q)^\nu + g^{\mu\nu}(-k-2q)\cdot(k+2q)\Big]$$

Simplify

$$g^{\mu\nu}(-q^2+2q\cdot k-k^2-k^2-4q^2-4q\cdot k)$$
$$+ (2k+q)^\mu (k^\nu - q^\nu - 2dk^\nu - dq^\nu + k^\nu + 2q^\nu)$$
$$+ (q-k)^\mu (-2k^\nu - q^\nu + k^\nu + 2q^\nu) + (k+2q^\mu)(2k^\nu + q^\nu + q^\nu - k^\nu)$$
$$= g^{\mu\nu}(-5q^2 - 2q\cdot k - 2k^2) - (2k+q)^\mu (2k+q)^\nu d$$
$$+ (2k+q)^\mu (2k+q)^\nu + (q-k)^\mu (q-k)^\nu + (k+2q)^\mu (k+2q)^\nu$$
$$= -g^{\mu\nu}(5q^2 + 2q\cdot k + 2k^2)$$
$$- d(q+2k)^\mu (q+2k)^\nu$$
$$+ 6k^\mu k^\nu + 3k^\mu q^\nu + 3q^\mu k^\nu + 6q^\mu q^\nu$$

Shift numerator $k = \ell - xq$

Drop terms linear in ℓ.

$$= -g^{\mu\nu}(5q^2 - 2xq^2 + 2\ell^2 + 2x^2 q^2)$$
$$- d(q^\mu q^\nu + 4\ell^\mu \ell^\nu + 4x^2 q^\mu q^\nu - 2xq^\mu q^\nu - 2xq^\mu q^\nu)$$
$$+ 6\ell^\mu \ell^\nu + 6x^2 q^\mu q^\nu - 3xq^\mu q^\nu - 3xq^\mu q^\nu + 6q^\mu q^\nu$$

Use $\ell^\mu \ell^\nu = \frac{1}{d}\ell^2 g^{\mu\nu}$

$$= -g^{\mu\nu}(6\ell^2)(1-\tfrac{1}{d})$$
$$-g^{\mu\nu}(5-2x+2x^2)q^2$$
$$- d(1-4x+4x^2)q^\mu q^\nu + 6q^\mu q^\nu(1-x+x^2)$$

Need two dim. reg. integrals.

$$\int \frac{d^d\ell}{(2\pi)^d} \frac{1}{(\ell^2-\Delta)^n} = \frac{(-1)^n i}{(4\pi)^{d/2}} \frac{\Gamma(n-d/2)}{\Gamma(n)} \left(\frac{1}{\Delta}\right)^{n-d/2}$$

$$\int \frac{d^d\ell}{(2\pi)^d} \frac{\ell^2}{(\ell^2-\Delta)^n} = \frac{(-1)^{n-1} i}{(4\pi)^{d/2}} \frac{d}{2} \frac{\Gamma(n-\frac{d}{2}-1)}{\Gamma(n)} \left(\frac{1}{\Delta}\right)^{n-\frac{d}{2}-1}$$

$$\Pi_{\mu\nu} = \frac{-1}{2} g^2 C_2(G)\delta^{ab} \times \int dx$$
$$\left[-6g^{\mu\nu}(1-\tfrac{1}{d})\left(\frac{-i}{(4\pi)^{d/2}} \frac{d}{2} \frac{\Gamma(1-d/2)}{\Gamma(2)} \left(\frac{1}{\Delta}\right)^{1-d/2}\right)\right.$$
$$+ \left[-g^{\mu\nu}q^2(5-2x+2x^2) - dq^\mu q^\nu(1-4x+4x^2)\right.$$
$$\left. + 6q^\mu q^\nu(1-x+x^2)\right]\left(\frac{i}{(4\pi)^{d/2}} \frac{\Gamma(2-d/2)}{\Gamma(2)} \left(\frac{1}{\Delta}\right)^{2-d/2}\right)$$

$$= -\frac{1}{2} g^2 C_2(G)\, \delta^{ab} \int dx$$

$$\Big[\, 6 g^{\mu\nu} \left(\frac{d}{2} - \frac{1}{2}\right) \frac{i}{(4\pi)^{d/2}} \frac{\Gamma(1-d/2)}{(\Delta)^{1-d/2}}$$

$$+ \left(g^{\mu\nu} q^2(-5+2x-2x^2) + q^\mu q^\nu\left(-d(1-4x+4x^2) + 6(1-x+x^2)\right)\right)$$

$$\frac{i}{(4\pi)^{d/2}} \frac{\Gamma(2-d/2)}{\Delta^{2-d/2}} \Big]$$

Group common factors $\frac{i}{(4\pi)^{d/2}} \left(\frac{1}{\Delta}\right)^{2-d/2}$, $\Delta = x(x-1)q^2$

$$= -\frac{1}{2} g^2 C_2(G)\, \delta^{ab} \int dx$$

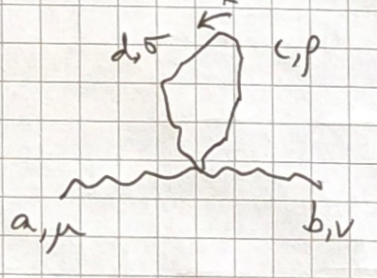
$$\Big[\, 3 g^{\mu\nu} (d-1)\, \Gamma(1-d/2)\, x(x-1) q^2$$

$$+ \left(g^{\mu\nu} q^2(-5+2x-2x^2) + q^\mu q^\nu\left(-d(1-4x+4x^2) + 6(1-x+x^2)\right)\right)$$

$$\cdot \Gamma(2-d/2) \Big]$$

Note the appearance of $\Gamma(1-d/2)$ & $\Gamma(2-d/2)$, which have different poles as $d \to 4$. It is a clear sign that our calculation is incomplete, because we started from a renormalizeable & should only need a finite number of counterterms to absorb the divergence. But a given counterterm cannot absorb two different poles.

Need to add the remaining gauge & ghost diagrams!



symmetry factor

$$\Pi_{\mu\nu} = \frac{1}{2} \int \frac{d^4k}{(2\pi)^4} \frac{-i g_{\rho\sigma}}{k^2} \delta^{cd} (-ig^2)$$

$$\Big[f^{abe} f^{cde} (g^{\mu\rho} g^{\nu\sigma} - g^{\mu\sigma} g^{\nu\rho})$$

$$+ f^{ace} f^{bde} (g^{\mu\nu} g^{\rho\sigma} - g^{\mu\sigma} g^{\nu\rho})$$

$$+ f^{ade} f^{bce} (g^{\mu\nu} g^{\rho\sigma} - g^{\mu\rho} g^{\nu\sigma}) \Big]$$

$$= -g^2 C_2(G)\, \delta^{ab} \int \frac{d^4k}{(2\pi)^4} \frac{1}{k^2} g^{\mu\nu} (d-1)$$

In order to combine above, need to have same denominator!

$$= -g^2 C_2(G)\, \delta^{ab} \int \frac{d^4k}{(2\pi)^4} \frac{(q+k)^2}{k^2 (q+k)^2} g^{\mu\nu} (d-1)$$

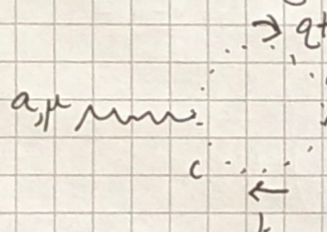
$$= -g^2 C_2(G)\,\delta^{ab}\int\frac{d^4\ell}{(2\pi)^4}\int_0^1 dx\,\frac{1}{(\ell^2-\Delta)^2}\,g^{\mu\nu}(d-1)\left(\ell^2+(1-x)^2q^2\right)$$

$$= \frac{ig^2}{(4\pi)^{d/2}}\,C_2(G)\,\delta^{ab}\int dx\,\left(\frac{1}{\Delta}\right)^{2-d/2}$$

Still have $\Gamma(1-d/2)$ + $\Gamma(2-d/2)$ poles!

$$\left[-\Gamma(1-d/2)\,g^{\mu\nu}q^2\left(\tfrac{1}{2}d(d-1)x(1-x)\right) - \Gamma(2-d/2)\,g^{\mu\nu}q^2(d-1)(1-x)^2\right]$$

Ghost diagram



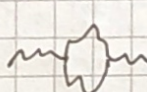
fermionic spin statistics

$$= (-1)\int\frac{d^4k}{(2\pi)^4}\,\frac{i}{k^2}\,\frac{i}{(k+q)^2}\,g^2 f^{dca}(k+q)^\mu f^{cbd}k^\nu$$

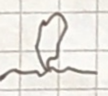
$$= \frac{ig^2}{(4\pi)^{d/2}}\,C_2(G)\,\delta^{ab}\int dx\,\left(\frac{1}{\Delta}\right)^{2-d/2}$$

$$\Big(-\Gamma(1-d/2)\,g^{\mu\nu}q^2\left(\tfrac{1}{2}x(1-x)\right) + \Gamma(2-\tfrac{d}{2})\,q^\mu q^\nu\left(x(1-x)\right)\Big)$$

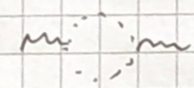
Now, combine worst divergence for gauge + ghost diagrams: $\Gamma(1-d/2)$ term



$$\frac{-1}{2}g^2 C_2(G)\,\delta^{ab}\int dx\,\frac{i}{(4\pi)^{d/2}}\left(\frac{1}{\Delta}\right)^{2-d/2} g^{\mu\nu}\,\Gamma(1-d/2)\,3(d-1)\,q^2x(x-1)$$



$$+\frac{i}{(4\pi)^{d/2}}\,g^2C_2(G)\,\delta^{ab}\int dx\,\left(\frac{1}{\Delta}\right)^{2-d/2}\left(-\Gamma(1-d/2)\,g^{\mu\nu}q^2\,\tfrac{1}{2}\,d(d-1)x(1-x)\right)$$



$$+\frac{ig^2}{(4\pi)^{d/2}}\,C_2(G)\,\delta^{ab}\int dx\,\left(\frac{1}{\Delta}\right)^{2-d/2}\left(-\Gamma(1-d/2)\,g^{\mu\nu}q^2\,\tfrac{1}{2}\,x(1-x)\right)$$

$$= \frac{ig^2}{(4\pi)^{d/2}}\,C_2(G)\,\delta^{ab}\int dx\,\left(\frac{1}{\Delta}\right)^{2-d/2} g^{\mu\nu}q^2\,x(1-x)$$

$$\left[\tfrac{1}{2}\cdot 3(d-1) - \tfrac{1}{2}d(d-1) - \tfrac{1}{2}\right]$$

Then $[\ldots] = (d-2)(1-d/2)$, where $(1-d/2)\,\Gamma(1-d/2) = \Gamma(2-d/2)$

and lifts the pole to the same place as the rest of the terms. Thus we recover gauge invariance + the theory is amenable to a counterterm.

Combine remaining terms:

$$\Pi_{\mu\nu} = \frac{-1}{2} g^2 C_2(G)\, \delta^{ab} \int dx \, \frac{i}{(4\pi)^{d/2}} \left(\frac{1}{\Delta}\right)^{2-d/2}$$

$$\Big(g^{\mu\nu} q^2 (-5 + 2x - 2x^2)\, \Gamma(2-d/2)$$

$$+ q^\mu q^\nu \big(d(-1+4x-4x^2) + 6(1-x+x^2)\big)\Gamma(2-d/2) \Big)$$

$$+ \frac{ig^2}{(4\pi)^{d/2}} C_2(G)\, \delta^{ab} \int dx \left(\frac{1}{\Delta}\right)^{2-d/2}$$

$$\left(-\Gamma(2-d/2)\, g^{\mu\nu} q^2 (d-1)(1-x)^2 \right)$$

$$+ \frac{ig^2}{(4\pi)^{d/2}} C_2(G)\, \delta^{ab} \int dx \left(\frac{1}{\Delta}\right)^{2-d/2}$$

$$\left(\Gamma(2-d/2)\, q^\mu q^\nu x(1-x) \right)$$

$$+ \frac{ig^2}{(4\pi)^{d/2}} C_2(G)\, \delta^{ab} \int dx \left(\frac{1}{\Delta}\right)^{2-d/2}$$

$$g^{\mu\nu}\, \Gamma(2-d/2)\, q^2 x(1-x)(d-2)$$

$$= \frac{ig^2}{(4\pi)^{d/2}} C_2(G)\, \delta^{ab} \int dx \, \frac{\Gamma(2-d/2)}{(\Delta)^{2-d/2}}$$

$$\Big[g^{\mu\nu} q^2 \big(\tfrac{-1}{2}(-5+2x-2x^2) - (d-1)(1-x)^2 + x(1-x)(d-2) \big)$$

$$+ q^\mu q^\nu \big(\tfrac{-1}{2}\big(d(-1+4x-4x^2) + 6(1-x+x^2)\big) + x(1-x)\big) \Big]$$

We know that these coefficients must match since the Ward identity enforces a $(g^{\mu\nu} q^2 - q^\mu q^\nu)$ structure. This matching is only true after integrating over x, though.

Use trick: Since Δ is symmetric for $x \leftrightarrow (1-x)$, we can replace

$$\int dx \, \frac{x}{(\Delta)^n} = \frac{1}{2}\int dx \, \frac{x}{\Delta^n} + \frac{1}{2}\int dx \, \frac{1-x}{\Delta^n} = \int dx \, \frac{1/2}{\Delta^n}$$

Then after simplification, we get

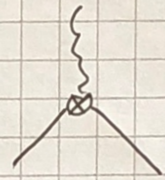
$$\Pi_{\mu\nu} = \frac{ig^2 C_2(G)\, \delta^{ab}}{(4\pi)^{d/2}} \int dx \left(\frac{1}{\Delta}\right)^{2-d/2} \Gamma(2-d/2)$$

$$(q^2 g^{\mu\nu} - q^\mu q^\nu)\left(1 + \frac{d}{2} + (4-2d)x^2 \right)$$

See Peskin + Schroeder 16.5 for a full discussion of counterterms + relations for each.

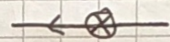
The discussion mirrors that of QED except that we have to take into account group factors / indices and also $\delta_1 \neq \delta_2$.

δ_1 : vertex ct.



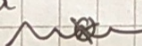
$ig\, t^a \gamma^\mu \delta_1$

δ_2 : fermion ct. propagator



$i\not{p}\, \delta_2$

δ_3 : gauge boson ct.

a  b

$\delta_3 (-i) (k^2 g^{\mu\nu} - k^\mu k^\nu) \delta^{ab}$

We set δ_3 by the divergence in the gauge, ghost, + fermion loops.

Recall

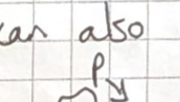
$$\Pi_{\mu\nu}^{\text{fermion}} = \sum_f \frac{ig^2}{16\pi^2} (q^2 g^{\mu\nu} - q^\mu q^\nu) \delta^{ab} \left(\frac{-4}{3} C(r_f) \right) \left(\frac{2}{\epsilon} \right)$$

Also, isolating divergence in $\Pi_{\mu\nu}^{\text{gauge+ghost}}$, we can integrate over dx to get

$$\Pi_{\mu\nu}^{\substack{\text{gauge+}\\ \text{ghost}}} = \frac{ig^2}{16\pi^2} C_2(G)\, \delta^{ab} (g^{\mu\nu} q^2 - q^\mu q^\nu) \left(\frac{5}{3} \right) \left(\frac{2}{\epsilon} \right)$$

Then, $$\delta_3 = \frac{g^2}{(4\pi)^2} \frac{\Gamma(2 - d/2)}{(M^2)^{2-d/2}} \left(\frac{5}{3} C_2(G) - \frac{4}{3} n_f C(r_f) \right)$$

We can also solve δ_2:

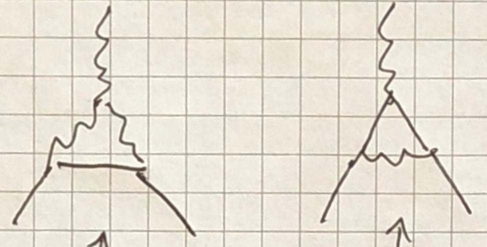


$$= \int \frac{d^4 p}{(2\pi)^4} (ig)^2 \gamma^\mu t^a \frac{i(\not{p} + \not{k})}{(p+k)^2} \gamma_\mu t^a \frac{-i}{p^2}$$

$$= \frac{ig^2}{(4\pi)^{d/2}} C_2(r_f)\, \not{k} \int dx\, (1-x)(d-2) \frac{\Gamma(2 - d/2)}{\Delta^{2-d/2}}$$

So $$\delta_2 = \frac{-g^2}{16\pi^2} \frac{\Gamma(2 - d/2)}{(M^2)^{2-d/2}} C_2(r_f)$$

Finally, for δ_1, we need the diagrams for the vertex correction at one-loop.



new in Y-M — same from QED

Get $$\delta_1 = \frac{-g^2}{16\pi^2} \frac{\Gamma(2-d/2)}{(M^2)^{2-d/2}} \left[C_2(r_f) + C_2(G)\right]$$

Now, $$\beta(e) = \mu\frac{\partial}{\partial\mu}\left(-e\delta_1 + e\delta_2 + \frac{e}{2}\delta_3\right) \quad \text{for QED}$$

$$\beta(g) = \mu\frac{\partial}{\partial\mu}\left(-g\delta_1 + g\delta_2 + \frac{g}{2}\delta_3\right) \quad \text{for Yang-Mills.}$$

Calculating, we get
$$\beta(g) = -g\,\mu\frac{\partial\delta_1}{\partial\mu} + g\,\mu\frac{\partial}{\partial\mu}\delta_2 + \frac{g}{2}\mu\frac{\partial}{\partial\mu}\delta_3$$

Identifying $\mu \equiv M$, we have

$$= -g\left(\frac{-g^2}{16\pi^2}\right)\left(\cancel{C_2(r_f)} + C_2(G)\right)\left(-(2-d/2)\frac{\Gamma(2-d/2)}{(M^2)^{3-d/2}}\right)\cdot 2M^2$$

same cancellation as QED

$$\cancel{+g\left(\frac{-g^2}{16\pi^2}\right)C_2(r_f)\left(-(2-d/2)\frac{\Gamma(2-d/2)}{(M^2)^{3-d/2}}\right)2M^2}$$

$$+\frac{g}{2}\left(\frac{g^2}{16\pi^2}\right)\left(\frac{5}{3}C_2(G) - \frac{4}{3}n_f C(r_f)\right)\left(-(2-d/2)\frac{\Gamma(2-d/2)}{(M^2)^{3-d/2}}\right)2M^2$$

$(2-d/2)\Gamma(2-d/2)$
$= \Gamma(3-d/2)$
$\xrightarrow{d\to 4} \Gamma(1) = 1$

$$= -\frac{g^3}{16\pi^2}C_2(G)\,2 - \frac{g^3}{16\pi^2}\left(\frac{5}{3}C_2(G) - \frac{4}{3}n_f C(r_f)\right)$$

$$= -\frac{g^3}{16\pi^2}\left(\frac{4}{3}C_2(G) - \frac{4}{3}n_f C(r_f)\right)$$

So, for Yang-Mills theory,

$$\beta(g) = \frac{-g^3}{16\pi^2}\left(\frac{11}{3} C_2(G) - \frac{4}{3} n_f C(r_f)\right)$$

at one-loop.

This is one of the most important results in field theory. Gross, Politzer, & Wilczek received the Nobel Prize in 2004 for this calculation. For n_f small, the β-fcn is negative & the coupling becomes weaker at higher energies. The theory is called asymptotically free. The gauge coupling becomes strong at low energies & theory exhibits confinement. For QCD, the scale of confinement is $\sim \Lambda_{QCD} = 200$ MeV.