Lecture 13.

QFT II.

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Last time: Non-Abelian Gauge Theory Lagrangian + functional method for quantization

Appearance of Fadeev-Popov ghosts

Today: Two main calculations to help interpret meaning of ghosts + their necessity in non-Abelian theory.

We will see that ghosts cancel unphysical gauge dofs. in loop calculations

In particular, ghosts have wrong spin-statistics to be external particles since they cannot be reps. of Poincáre / Lorentz symmetry.

First calculation will be $f\bar{f} \to G^a G^b$, second is the WFR / vacuum polarization diagram in non-Abelian theory.

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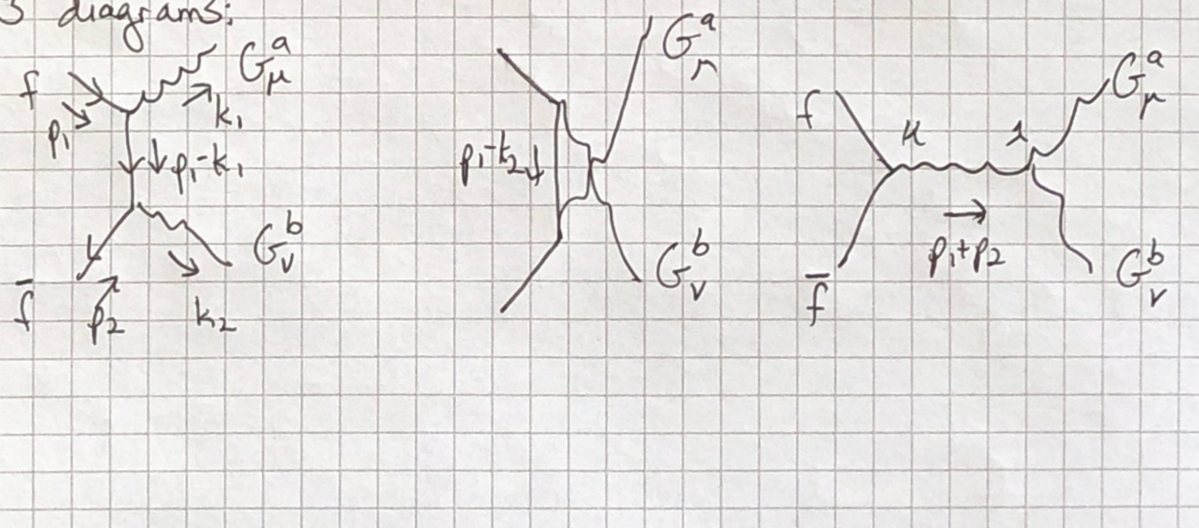
Consider $f\bar{f} \to G^a G^b$

3 diagrams:



Consider two fermion diagrams:

$$i\mathcal{M}_f = \bar{v}(p_2)(igt^b\gamma^\nu)\frac{i(\not{p}_1-\not{k}_1+m)}{(p_1-k_1)^2-m^2}igt^a\gamma^\mu u(p_1)\epsilon^*_\mu(k_1)\epsilon^*_\nu(k_2)$$

$$+\bar{v}(p_2)(igt^a\gamma^\mu)\frac{i(\not{p}_1-\not{k}_2+m)}{(p_1-k_2)^2-m^2}igt^b\gamma^\nu u(p_1)\epsilon^*_\mu(k_1)\epsilon^*_\nu(k_2)$$

Study Ward identity: $\epsilon^*_\nu(k_2)\to k_2$, should have $\mathcal{M}_{tot}$ vanish.

$$= \bar{v}(p_2)(-ig^2)\epsilon^*_\mu(k_1)\Big[(t^bt^a)\,\not{k}_2\frac{(\not{p}_1-\not{k}_1+m)\gamma^\mu}{(p_1-k_1)^2-m^2}$$

$$+(t^at^b)\frac{(\not{p}_1-\not{k}_2+m)}{(p_1-k_2)^2-m^2}\not{k}_2\Big]u(p_1)$$

Note $t^a$ + $t^b$ group generators act on group space indices + independent of Lorentz / Dirac structure.

Also, $p_1+p_2=k_1+k_2$, and

Dirac eqns. $\bar{v}(p_2)(\not{p}_2+m)=0$, $(\not{p}_1-m)u(p_1)=0$.

$$= (-ig^2)\epsilon^*_\mu(k_1)\bar{v}(p_2)\Big[(t^bt^a)\frac{(\not{k}_2-\not{p}_2-m)(\not{p}_1-\not{k}_1+m)\gamma^\mu}{(p_1-k_1)^2-m^2}$$

$$+(t^at^b)\frac{(\not{p}_1-\not{k}_2+m)(\not{k}_2-\not{p}_1+m)}{(p_1-k_2)^2-m^2}\Big]u(p_1)$$

Propagators cancel: $(\not{k}_2-\not{p}_2-m)(\not{p}_1-\not{k}_1+m) = (\not{p}_1-\not{k}_1-m)(\not{p}_1-\not{k}_1+m)$

$$= (+ig^2)\epsilon^*_\mu(k_1)\,\bar{v}(p_2)\gamma^\mu u(p_1)\,[t^a,t^b] \qquad = (p_1-k_1)^2-m^2$$

Use $[t^a,t^b]=if^{abc}t^c$

$$= -g^2\epsilon^*_\mu(k_1)\,\bar{v}(p_2)\gamma^\mu u(p_1)f^{abc}t^c$$

Gauge diagram. Choose $\xi = 1$ for gauge boson propagator.

$$i\mathcal{M}_V = \bar{v}(p_2)\,(ig\,t^c\gamma^\kappa)\,u(p_1)\,\epsilon^*_\mu(k_1)\,\epsilon^*_\nu(k_2)$$
$$\frac{-i}{(p_1+p_2)^2}\, g_{\kappa\lambda}\; g f^{abc}\left[g^{\mu\nu}(-k_1+k_2)^\lambda + g^{\nu\lambda}(-k_2-p_1-p_2)^\mu + g^{\lambda\mu}(p_1+p_2+k_1)^\nu\right]$$

Again, use $\epsilon^*_\nu(k_2) \to k_{2\nu}$

$$= \frac{g^2 f^{abc} t^c}{(p_1+p_2)^2}\,\bar{v}(p_2)\gamma^\kappa u(p_1)\,\epsilon^*_\mu(k_1)$$
$$\left[k_2^\mu(-k_1+k_2)^\lambda + k_2^\lambda(-k_2-k_1-k_2)^\mu + g^{\mu\lambda}(2k_1+k_2)\cdot k_2\right]$$

$$= \frac{g^2 f^{abc} t^c}{(p_1+p_2)^2}\Big[\bar{v}(p_2)\,(\not{k}_1+\not{k}_2)\,u(p_1)\,(-k_2\cdot\epsilon^*(k_1))$$
$$-\bar{v}(p_2)\,\not{k}_2\,u(p_1)\;k_1\cdot\epsilon^*(k_1)$$
$$+\,2\,k_1\cdot k_2\;\bar{v}(p_2)\gamma^\mu u(p_1)\Big)$$

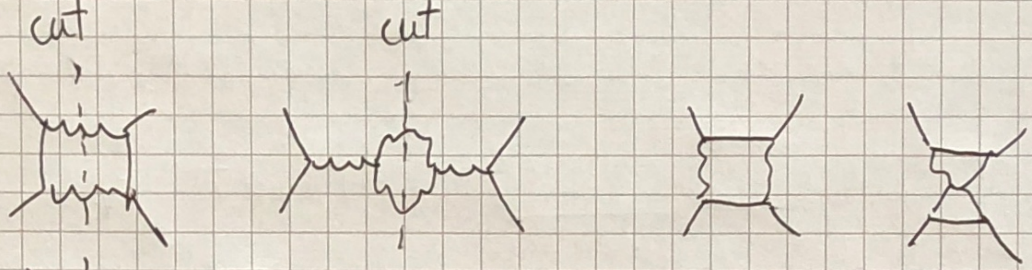
Note $\bar{v}(p_2)[\not{k}_1+\not{k}_2]\,u(p_1) = \bar{v}(p_2)\,(\not{p}_2+m+\not{p}_1-m)\,u(p_1) = 0$

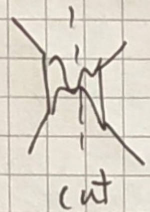
Also, the third line cancels $i\mathcal{M}_f$, since
$$2k_1\cdot k_2 = (k_1+k_2)^2 = (p_1+p_2)^2.$$

$$i\mathcal{M}_{tot} = \frac{g^2 f^{abc} t^c}{(p_1+p_2)^2}\left(-\bar{v}(p_2)\,\not{k}_2\,u(p_1)\;k_1\cdot\epsilon^*(k_1)\right)$$

We need this to vanish in order to preserve Ward identity. Fortunately, for on-shell gauge bosons, $k_1\cdot\epsilon^*(k_1)=0$ since the gauge boson is transverse. But this is not entirely satisfactory:

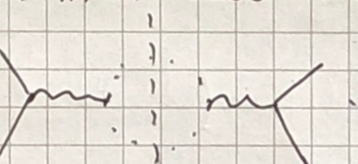
Consider the tree-loop relation provided by the optical theorem. The $f\bar{f}\to G^a G^b$ tree-level process is then the cut version of $f\bar{f}\to\xi\, f\bar{f}$ at 1-loop:



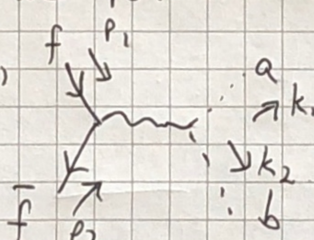




Cutkosky rule: Replace cut propagator by $(-ig_{\mu\nu})\,(-2\pi i\,\delta(k_i^2))$.

If we were to calculate the full 1-loop $\mathcal{M}$, we would have a non-zero $\mathrm{Im}\,\mathcal{M}$, since the intermediate gauge bosons are virtual and off-shell.
But then there are no tree-level diagrams that $\mathrm{Im}\,\mathcal{M}_{1\text{-loop}}$ would correspond to, leading to a violation of unitarity.
This is cured by the fact that the 1-loop diagram must also include the ghost diagram:



If we perform the same cut & evaluate the tree diagram,




$$i\mathcal{M}_{ghost} = \bar v(p_2)\; i g\, t^c \gamma^\kappa u(p_1)\; \frac{-i g_{\kappa\lambda}}{(p_1+p_2)^2}\;(-g f^{acb} k_1^\lambda)$$

$$= \bar v(p_2)\; g^2 \frac{f^{abc} t^c \gamma^\kappa}{(p_1+p_2)^2} u(p_1)\, k_1^\kappa$$

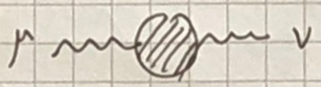
& also for the crossed diagram. This cancels the gauge bosons in the loop: Use $k_1 = (k^0, \vec k)$

$$\epsilon^+_\mu(k) = \left(\frac{k^0}{\sqrt2|\vec k|}, \frac{\vec k}{\sqrt2|\vec k|}\right), \quad \epsilon^-_\mu(k) = \left(\frac{k^0}{\sqrt2|\vec k|}, \frac{-\vec k}{\sqrt2|\vec k|}\right)$$

The $(\mathcal{M}^{\mu\nu}\epsilon^{-*}_\mu \epsilon^{+*}_\nu)(\mathcal{M}^{\rho\sigma})(\epsilon^-_\rho \epsilon^+_\sigma)$ piece does not vanish.

Second calculation

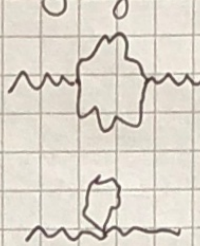
Vacuum polarization $\Pi_{\mu\nu}(q^2)$



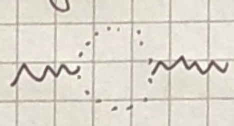
At one-loop, we have 3 (4) types of diagrams:

fermion loop

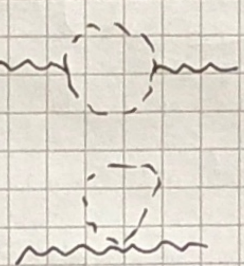
gauge



ghost



scalar loops (if present in the theory)



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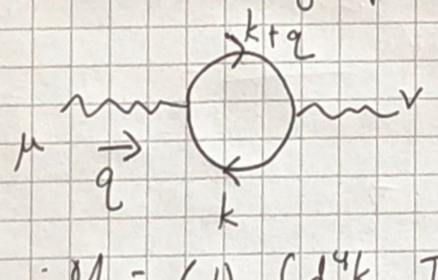
Start with fermion loop.

Will be same as QED vacuum polarization except for the additional group factor matrices.



$$i\mathcal{M} = (-1)\int \frac{d^4k}{(2\pi)^4}\,\mathrm{Tr}\left[\frac{i(\not k+m)}{(k^2-m^2)}\, i g t^b \gamma^\nu \,\frac{i(\not k+\not q+m)}{(k+q)^2-m^2}\, i g t^a \gamma^\mu\right]$$

$$= -g^2\int \frac{d^4k}{(2\pi)^4}\,\mathrm{Tr}[t^b t^a]\,\frac{1}{k^2-m^2}\,\frac{1}{(k+q)^2-m^2}\,\mathrm{Tr}\left[\not k\gamma^\nu(\not k+\not q)\gamma^\mu + m^2\gamma^\nu\gamma^\mu\right]$$

$$= -4g^2\,\mathrm{Tr}[t^a t^b]\int\frac{d^4k}{(2\pi)^4}\int_0^1 dx\,\frac{1}{(k^2+2xk\cdot q+xq^2-m^2)^2}\,\big(k^\nu(k+q)^\mu - g^{\mu\nu}k\cdot(k+q) +$$

numerator continued $$+\,k^\mu(k+q)^\nu + m^2 g^{\mu\nu}\big)$$

Shift $l \equiv k + xq$

Numerator is:

$$\left(2k^\mu k^\nu + k^\mu q^\nu + q^\mu k^\nu - g^{\mu\nu}(k^2 + k\cdot q) + m^2 g^{\mu\nu}\right)$$

Only keep $l^2$ quadratic + $l^0$ powers

$$= 2l^\mu l^\nu + 2x^2 q^\mu q^\nu - xq^\mu q^\nu - xq^\mu q^\nu - g^{\mu\nu}(l^2 + x^2q^2 - xq^2) + m^2 g^{\mu\nu}$$

Denominator

$$l^2 - x^2q^2 + xq^2 - m^2 = l^2 - \Delta$$

for $\Delta \equiv m^2 + x^2q^2 - xq^2$

We get

$$= -4g^2 C(r)\delta^{ab}\int \frac{d^dl}{(2\pi)^d}\int_0^1 dx \frac{1}{(l^2-\Delta)^2}\Big[\left(2l^\mu l^\nu - l^2 g^{\mu\nu}\right) + 2x^2q^\mu q^\nu - 2xq^\mu q^\nu + g^{\mu\nu}(-x^2q^2 + xq^2 + m^2)\Big]$$

Use $\int \frac{d^dl}{(2\pi)^d}\frac{2l^\mu l^\nu - l^2 g^{\mu\nu}}{(l^2-\Delta)^2} = \int \frac{d^dl}{(2\pi)^d}\frac{\left(\frac{2}{d}-1\right)l^2 g^{\mu\nu}}{(l^2-\Delta)^2}$

$$= \left(\frac{2}{d}-1\right)g^{\mu\nu}\cdot\left(\frac{-1 \cdot i}{(4\pi)^{d/2}}\left(\frac{d}{2}\right)\frac{\Gamma(1-\frac{d}{2})}{\Gamma(2)}\left(\frac{1}{\Delta}\right)^{1-\frac{d}{2}}\right)$$

$$= -\frac{i g^{\mu\nu}}{(4\pi)^{d/2}}\frac{\Gamma(2-d/2)}{(\Delta)^{2-d/2}}\,\Delta$$

$\Delta = m^2 + x^2q^2 - xq^2$

$$\int \frac{d^dl}{(2\pi)^d}\frac{1}{(l^2-\Delta)^2} = \frac{i}{(4\pi)^{d/2}}\frac{\Gamma(2-d/2)}{(\Delta)^{2-d/2}}$$

$$= -4g^2C(r)\delta^{ab}\int_0^1 dx\left[\frac{i}{(4\pi)^{d/2}}\frac{\Gamma(2-d/2)}{\Delta^{2-d/2}}\right]$$

$$\Big(-g^{\mu\nu}(\cancel{m^2} + x^2q^2 - xq^2) + 2x(x-1)q^\mu q^\nu + g^{\mu\nu}(\cancel{m^2} - x^2q^2 + xq^2)\Big)$$

$$= -4g^2C(r)\delta^{ab}\int_0^1 dx \frac{i}{(4\pi)^{d/2}}\frac{\Gamma(2-d/2)}{(\Delta)^{2-d/2}}$$

$$\left(2x(1-x)\right)\left(q^2g^{\mu\nu} - q^\mu q^\nu\right)$$

$q^2 g^{\mu\nu} - q^\mu q^\nu$ structure guaranteed by WI

We get

$$= i g^2 C(r) \delta^{ab} (q^2 g^{\mu\nu} - q^\mu q^\nu) \int_0^1 dx \, \frac{8x(x-1)}{(4\pi)^{d/2}} \frac{\Gamma(2-d/2)}{\Delta^{2-d/2}}$$

For divergent piece, $\frac{1}{(4\pi)^{d/2}} \frac{\Gamma(2-d/2)}{\Delta^{2-d/2}} \rightarrow \frac{1}{16\pi^2} \cdot \frac{2}{\epsilon}$

and $\int_0^1 dx \; 8x^2 - 8x = \frac{8}{3} - 4 = -\frac{4}{3}$

$$= \frac{ig^2}{16\pi^2} (q^2 g^{\mu\nu} - q^\mu q^\nu) C(r) \delta^{ab} \left(-\frac{4}{3}\right)\left(\frac{2}{\epsilon}\right)$$

For $N_F$ flavors in reps. $r_f$,

$$= \sum_f \frac{ig^2}{16\pi^2} (q^2 g^{\mu\nu} - q^\mu q^\nu) \delta^{ab} \left(\frac{-8}{3\epsilon} C(r_f)\right)$$

Recall $C(\square) = C(\text{"fundamental"}) = \frac{1}{2}$.