Lecture 12.

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QFT II.

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Last time: Group theory + representation theory

Today: Non-Abelian Gauge Theory.

Terminology:

Non-Abelian theory refers to (typically) $SU(N)$ gauge theory.

Yang-Mills: specifically only gauge fields, no fermions.

$$\mathcal{L} = -\frac{1}{4} F^a_{\mu\nu} F^{a\mu\nu}$$

QCD or QCD-like: $SU(N_c)$ theory with N_F flavors of massive / massless quarks in fundamental reps.

Quarks have vector masses, not chiral.

Will study non-Abelian theory with one fermion

$$\mathcal{L} = -\frac{1}{4}(G^a_{\mu\nu})^2 + \bar{\Psi}(i\not{D} - m)\Psi$$

$$D_\mu = \partial_\mu - ig A^a_\mu t^a$$

$$G^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^{abc} A^b_\mu A^c_\nu$$

Non-linear terms:

$$\mathcal{L} = -\frac{1}{4}(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^{abc} A^b_\mu A^c_\nu)^2 + \bar{\Psi}(i\not{D} - m)\Psi$$

$$= \mathcal{L}_{free} + g A^a_\lambda \bar{\Psi} \gamma^\lambda t^a \Psi - g f^{abc}(\partial_\kappa A^a_\lambda) A^{\kappa b} A^{\lambda c}$$

$$-\frac{1}{4} g^2 (f^{eab} A^a_\kappa A^b_\lambda)(f^{ecd} A^{\kappa c} A^{\lambda d})$$

Fermion propagator: $\dfrac{i}{\not{k} - m}\delta_{ij}$ for $\langle \Psi_i \bar{\Psi}_j \rangle$

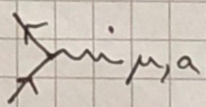
$i \longleftarrow j$

color indices

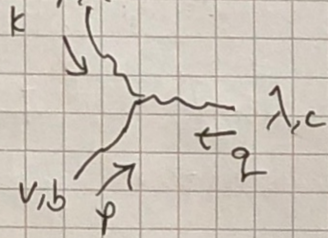
Massless vector: $\dfrac{-i g_{\mu\nu}}{k^2}\delta^{ab}$ for $\langle A^a_\mu A^b_\nu \rangle$

a ~~~~ b

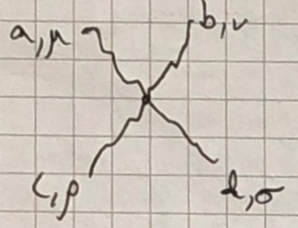
Can easily derive



$ig t^a \gamma^\mu$



$$g f^{abc}\left[g^{\mu\nu}(k-p)^\lambda + g^{\nu\lambda}(p-q)^\mu + g^{\lambda\mu}(q-k)^\nu\right]$$



$$-ig^2\left[f^{abe}f^{cde}\left(g^{\mu\rho}g^{\nu\sigma}-g^{\mu\sigma}g^{\nu\rho}\right) + f^{ace}f^{bde}\left(g^{\mu\nu}g^{\rho\sigma}-g^{\mu\sigma}g^{\nu\rho}\right) + f^{ade}f^{bce}\left(g^{\mu\nu}g^{\rho\sigma}-g^{\mu\rho}g^{\nu\sigma}\right)\right]$$

[Important: different powers of same gauge coupling in 3 pt. + 4 pt. vertices. Critical for overall gauge invariance: must draw $\underline{\text{all}}$ diagrams at given $\mathcal{O}(g^n)$ and loop order $\left(\frac{1}{16\pi^2}\right)$ to get gauge invariant result.

Now, quantize Abelian + non-Abelian theory.

P+S 9.4

Recall origin of need for gauge-fixing. In Abelian,

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar\Psi i\not{\partial}\Psi$$

or

$$-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + J_\mu A^\mu \quad \text{for many matter fields charged under } U(1).$$

Then EOM is $\left(k^2 g_{\mu\nu} - k_\mu k_\nu\right)A_\nu = J_\mu$

Note
$$S = \int d^4x\left(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}\right)$$
$$= \frac{1}{2}\int d^4x\, A_\mu(x)\left(\partial^2 g^{\mu\nu} - \partial^\mu\partial^\nu\right)A_\nu(x)$$
$$= \frac{1}{2}\int \frac{d^4k}{(2\pi)^4}\,\tilde{A}_\mu(k)\left(-k^2 g^{\mu\nu} + k^\mu k^\nu\right)\tilde{A}_\nu(-k)$$

Not invertible because operator has det = 0 from k_μ eigenvector.
Cannot uniquely solve for A_μ given J_μ, because $A_\mu \to A_\mu + \frac{1}{e}\partial_\mu\alpha$ has same EOM.

In the fcn'al integral,

$$\int \mathcal{D}A\, e^{iS[A]} \equiv \int \mathcal{D}A^0 \mathcal{D}A^1 \mathcal{D}A^2 \mathcal{D}A^3\, e^{iS[A]},$$

the field configurations that are gauge-equivalent to 0 give an infinite redundancy. Goal is to modify $S[A]$ to calculate and cancel this redundancy in observables. This procedure is called the Fadeev-Popov method.

Consider $G(A) = \partial^\mu A_\mu(x) - \omega(x)$, $\omega(x)$ is any scalar fcn.
Then $G(A^\alpha) = G(A_\mu + \frac{1}{e}\partial_\mu \alpha)$, $G(A)$ is fcn. that we wish to set equal to zero as a gauge fixing condition, so Lorentz gauge is $G(A) = 0 = \partial^\mu A_\mu(x)$
$\Rightarrow \omega(x) = 0$. Then, $G(A^\alpha) = \partial^\mu A_\mu + \frac{1}{e}\partial^2 \alpha$.
Use functional delta fcn., $\delta(G(A))$ to constrain fcn'al integral to cover only configurations with $G(A) = 0$.

We will need to identity

$$1 = \int \mathcal{D}\alpha(x)\, \delta(G(A^\alpha)) \det\left(\frac{\delta G(A^\alpha)}{\delta \alpha}\right)$$

↳ inf. product of δ-fcns. for each pt. x.

Justification:

$$\int_{-\infty}^{\infty} f(x)\, \delta(g(x))\, dx = \sum_i \frac{f(x_i)}{|g'(x_i)|} \quad \text{for } x_i \text{ as roots of } g(x)$$

where $\frac{1}{|g'(x_i)|}$ comes from Jacobian for changing arg. x to $g(x)$. Analogously,

$$1 = \left(\prod_i \int da_i\right) \delta^{(n)}(g(a)) \det\left(\frac{\partial g_i}{\partial a_j}\right)$$

so change a_i to $g(a)$ requires Jacobian. In continuum, $\left(\prod_i \int da_i\right) \to \int \mathcal{D}a$ as functional integral.

Using this identity inside $\int \mathcal{D}A\, e^{iS[A]}$

$$= \int \mathcal{D}A\, \mathcal{D}\alpha\ \delta(G(A^\alpha)) \det\left(\frac{\delta G(A^\alpha)}{\delta\alpha}\right) e^{iS[A]}$$

Remark: in Abelian theory, $\det\left(\frac{\delta G(A^\alpha)}{\delta \alpha}\right) = \det\left(\frac{\partial^2}{e}\right)$ is independent of α and A.

Since $S[A]$ is gauge invariant, we can perform a gauge transformation to $S[A^\alpha]$ and then $\mathcal{D}A = \mathcal{D}A^\alpha$, but we can then rename to A + there is no α dependence.

$$= \det\left(\frac{\delta G(A^\alpha)}{\delta\alpha}\right) \left(\int \mathcal{D}\alpha\right) \int \mathcal{D}A\, e^{iS[A]}\, \delta(G[A])$$

so the functional integral $\int \mathcal{D}A$ is now restricted to physically inequivalent field configurations. (If $G(A) \neq 0$, integral vanishes.)

For $G(A) = \partial_\mu A^\mu - \omega$,

$$\int \mathcal{D}A\, e^{iS[A]} = \det\left(\frac{1}{e}\partial^2\right) \left(\int \mathcal{D}\alpha\right) \int \mathcal{D}A\, e^{iS[A]}\, \delta(\partial^\mu A_\mu - \omega(x))$$

for any ω, so linearly combine different ω with appropriate normalization. Introduce integration over ω:

$$N(\xi) \int \mathcal{D}\omega\, \exp\left[-i\int d^4x\, \frac{\omega^2}{2\xi}\right] \det\left(\frac{1}{e}\partial^2\right)\left(\int \mathcal{D}\alpha\right) \int \mathcal{D}A\, e^{iS[A]}\, \delta(\partial^\mu A_\mu - \omega).$$

$$= \underbrace{N(\xi)\ \det\left(\frac{1}{e}\partial^2\right) \left(\int \mathcal{D}\alpha\right)}_{\text{cancels in all corr. fcns}} \int \mathcal{D}A\, e^{iS[A]} \exp\Big[\underbrace{-i\int d^4x\, \frac{1}{2\xi}(\partial^\mu A_\mu)^2}_{\text{new contribution to } \mathcal{L}.}\Big]$$

In a time-ordered product of corr. fcn. of A,

$$\langle \Omega | \, T \, \mathcal{O}(A) \, | \Omega \rangle =$$

$$\lim_{T \to \infty(1-i\epsilon)} \frac{\int \mathcal{D}A \; \mathcal{O}(A) \exp\left[i \int_{-T}^{T} d^4x \, \mathcal{L} \right]}{\int \mathcal{D}A \, \exp\left[i \int_{-T}^{T} d^4x \, \mathcal{L} \right]}$$

From Fadeev-Popov procedure, as long as $\mathcal{O}(A)$ is gauge invariant,

$$= \lim_{T \to \infty(1-i\epsilon)} \frac{\int \mathcal{D}A \; \mathcal{O}(A) \exp\left[i \int_{-T}^{T} d^4x \left(\mathcal{L} - \frac{1}{2\xi} (\partial^\mu A_\mu)^2 \right) \right]}{\int \mathcal{D}A \, \exp\left[i \int_{-T}^{T} d^4x \left(\mathcal{L} - \frac{1}{2\xi} (\partial^\mu A_\mu)^2 \right) \right]}$$

Now, propagator is ξ-dependent.

Get

$$\mu \sim\sim\sim \nu \qquad \frac{-i}{k^2 + i\epsilon} \left(g^{\mu\nu} - (1-\xi) \frac{k^\mu k^\nu}{k^2} \right)$$

$\xi = 0$ Landau gauge

$\xi = 1$ Feynman gauge.

$\xi = 3$ Yennie gauge.

Back to non-Abelian theory.

P+S, 16.2. Recall A^α is gauge field transformed through finite gauge trans.

We have

$$1 = \int \mathcal{D}\alpha(x) \, \delta(G(A^\alpha)) \det\left(\frac{\delta G(A^\alpha)}{\delta \alpha} \right)$$

has $(A^\alpha)^a_\mu t^a = e^{i\alpha^a t^a} \left(A^b_\mu t^b + \frac{i}{g} \partial_\mu \right) e^{-i\alpha^c t^c}$

or, if we consider α infinitesimal,

$$(A^\alpha)^a_\mu = A^a_\mu + \frac{1}{g} \partial_\mu \alpha^a + f^{abc} A^b_\mu \alpha^c$$

$$= A^a_\mu + \frac{1}{g} D_\mu \alpha^a$$ where D_μ is covariant derivative of α^a trans. in adjoint rep.

p.6

But $\det\left(\frac{\delta G(A^\alpha)}{\delta\alpha}\right) = \det\left(\frac{1}{g}\partial^\mu D_\mu\right)$ is no longer independent of A and so not simply a normalization factor.
Instead, interpret det as functional integral over new set of <u>anticommuting</u> scalar fields in <u>adjoint</u> <u>rep</u>.

Recall

$$\left(\prod_i \int d\theta_i^* d\theta_i\right) e^{-i\,\theta_i^* B_{ij}\theta_j} = \left(\prod_i \int d\theta_i^* d\theta_i\right) e^{-\sum_i \theta_i^* b_i \theta_i}$$

$$= \prod_i b_i = \det B$$

Using the logic in reverse, the $\det\left(\frac{1}{g}\partial^\mu D_\mu\right)$ operator can be introduced as a Lagrangian term

$$\det\left(\frac{1}{g}\partial^\mu D_\mu\right) = \int \mathcal{D}c\,\mathcal{D}\bar{c}\, \exp\left[i\int d^4x \left(\bar{c}\,(-\partial^\mu D_\mu)\,c\right)\right]$$

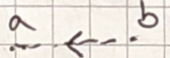
with $\frac{1}{g}$ in renormalization of $c, \bar{c}$.

The scalars are Fadeev-Popov ghosts c and anti-ghosts $\bar{c}$.

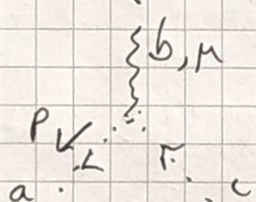
Get new Lagrangian term:

$$\mathcal{L}_{ghost} = \bar{c}^a\left(-\partial^2\delta^{ac} - g\,\partial^\mu f^{abc} A_\mu^b\right)c^c.$$

Ghosts are dots.



$= \frac{i\delta^{ab}}{p^2}$



$= -g f^{abc} p^\mu$

The non-Abelian propagator is same as before:

$$\frac{-i}{k^2+i\epsilon}\left(g^{\mu\nu} - (1-\xi)\frac{k^\mu k^\nu}{k^2}\right)$$

Rest of Fadeev-Popov procedure same: shift A to A^α, relabel, Gaussian integrate over ω, get ξ-dependent $\frac{1}{2\xi}(\partial^\mu A_\mu)^2$ in $\mathcal{L}$.