Lecture 10.

QFT II.

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Last time: Functional methods for scalar + spinor theories

Today: Begin unit on non-Abelian gauge theories

Subtopics: Group theory, Lie Algebras
Quantization and Fadeev-Popov ghosts
Quantum Chromodynamics

Refs. Georgi - Lie Algebras in Particle Physics
Wu-Ki Tung - Group Theory in Physics

From last time, we saw that fcnal. methods readily reproduce the known results of n-pt. correlation fcns. in free field theory. Then, a perturbative expansion of the action wrt. a perturbative coupling leads to the appropriate Feynman rule, where the fcnal derivatives on the generating fcnal, extract the appropriate external legs from the vacuum.

The last issue is the functional method for vector fields, which involves extra complications mainly because of the issue of gauge redundancy. We will visit this topic in detail in a few lectures, after we discuss + introduce non-Abelian gauge symmetries.

We first review Abelian gauge symmetries.

Recall QED:

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \bar{\Psi}(i\not{\partial} - m)\Psi + \bar{\Psi} e \gamma^\mu A_\mu \Psi$$

Permits a <u>local</u> symmetry transformation.

$$\psi(x) \to e^{i\alpha(x)}\psi(x)$$

If $\alpha(x) \equiv \alpha = \text{const.}$, this is a global phase rotation.

Define derivative $n^\mu \partial_\mu \psi(x) = \lim_{\epsilon\to 0} \frac{1}{\epsilon}\left[\psi(x+\epsilon n) - \psi(x)\right]$

We require a compensator (also called comparator) since two spacetime points are involved.

$$U(y,x) \underset{\substack{\text{transforms}\\ \text{as}}}{\to} e^{i\alpha(y)}\, U(y,x)\, e^{-i\alpha(x)}$$

Use $U(x,x) = 1$, then $U(y,x) = \exp[i\phi(y,x)]$ because we require $U(y,x)$ to be a pure phase, with $\phi(y,x)$ some scalar fcn. of y + x.

Thus, $\psi(y)$ and $U(y,x)\psi(x)$ transform the same way.

$\Rightarrow$ can define a <u>covariant</u> <u>derivative</u> based on subtracting the two: $n^\mu D_\mu \psi(x) = \lim_{\epsilon\to 0} \frac{1}{\epsilon}\left[\psi(x+\epsilon n) - U(x+\epsilon n, x)\psi(x)\right]$

Expand $U(x+\epsilon n, x) \approx 1 - ie\epsilon n^\mu A_\mu(x) + \mathcal{O}(\epsilon^2)$

$A_\mu(x)$ is <u>connection</u>

Then $D_\mu \psi(x) = \partial_\mu \psi(x) + ie A_\mu \psi(x)$

+ $A_\mu(x)$ transforms as $A_\mu \to A_\mu - \frac{1}{e}\partial_\mu \alpha(x)$

Derive A_μ transformation:

$$\begin{array}{ccc} U(y,x) & \to & e^{i\alpha(y)}\,(1 - ie\epsilon n^\mu A_\mu)\, e^{-i\alpha(x)} \\ \Downarrow & & \Downarrow \\ & & e^{i\alpha(x+\epsilon n)}\,(1 - ie\epsilon n^\mu A_\mu)\, e^{-i\alpha(x)} \end{array}$$

$$1 - ie\epsilon n^\mu A_\mu \to (1 + i\alpha(x+\epsilon n))(1 - ie\epsilon n^\mu A_\mu)(1 - i\alpha(x))$$

$$\Rightarrow 1 - ie\epsilon n^\mu A_\mu \to 1 + i\alpha(x+\epsilon n) - i\alpha(x) - ie\epsilon n^\mu A_\mu + \mathcal{O}(\epsilon^2))$$

At $\mathcal{O}(\epsilon)$:

$$-ie n^\mu A_\mu \to \frac{1}{\epsilon}\, i\left(\alpha(x+\epsilon n) - \alpha(x)\right) - ie n^\mu A_\mu$$

$$-ie n^\mu A_\mu \Rightarrow i\frac{\partial}{\partial x_\mu}\alpha(x) - ie n^\mu A_\mu$$

$$A_\mu \to A_\mu - \frac{1}{e}\frac{\partial \alpha(x)}{\partial x_\mu} \quad \checkmark$$

Then, $D_\mu\psi = \partial_\mu\psi + ieA_\mu\psi \to \left(\partial_\mu + ie\left(A_\mu - \frac{1}{e}\partial_\mu\alpha\right)\right)(e^{i\alpha}\psi)$

$= e^{i\alpha}\partial_\mu\psi + ieA_\mu e^{i\alpha}\psi + e^{i\alpha}(i\partial_\mu\alpha)\psi - i(\partial_\mu\alpha)e^{i\alpha}\psi$

$= e^{i\alpha}(D_\mu\psi)$

So, $D_\mu\psi$ transforms as ψ.

Then, $[D_\mu, D_\nu]\psi$ also transforms as ψ.

$[D_\mu, D_\nu]\psi = \underbrace{[\partial_\mu, \partial_\nu]}_{0 \text{ by commutation}}\psi + ie\left([\partial_\mu, A_\nu] - [\partial_\nu, A_\mu]\right)\psi - e^2\underbrace{[A_\mu, A_\nu]}_{0 \text{ by commutation}}\psi$

$= ie\,\partial_\mu(A_\nu\psi) + ie\,A_\mu\partial_\nu\psi$
$\quad - ie\,\partial_\nu(A_\mu\psi) - ie\,A_\nu\partial_\mu\psi$

$= ie(\partial_\mu A_\nu)\psi + ie\,A_\nu(\partial_\mu\psi) + ie\,A_\mu\partial_\nu\psi$
$\quad - ie(\partial_\nu A_\mu)\psi - ie\,A_\mu(\partial_\nu\psi) - ie\,A_\nu\partial_\mu\psi$

$= ie(\partial_\mu A_\nu - \partial_\nu A_\mu)\psi = ie\,F_{\mu\nu}\psi$

Since $\psi \to e^{i\alpha}\psi$ saturates the gauge transformation, we conclude that $F_{\mu\nu}$ is gauge invariant (in $U(1)$ theories.)

Now, we generalize the local symmetry to more complicated algebraic fcns.

First, consider a global symmetry.

This is known as a <u>flavor</u> <u>symmetry</u>, which is a very important aspect of SM physics and beyond.

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\psi}_i(i\not{\partial} - m_i)\psi_i + \bar{\psi}_i e Q_i \gamma^\mu A_\mu \psi_i$$

If each ψ_i has charge Q_i, then $\psi_i \to e^{iQ_i\alpha(x)}\psi_i$ under $U(1)$ transformations.

Note that if all $m_i = m$ and $Q_i = Q$, then the ψ_i fields are indistinguishable & can be rotated into each other, keeping the Lagrangian unchanged.

For N_f number of ψ_i fields, this rotation is a $U(N_f)$ transform.

$U(N_F) = U(1) \times SU(N_F)$.

For matrix $U \in SU(N_F)$, we would perform the rotation $U_{ij} \psi_j = \psi_i$.

Next, to make this a gauge symmetry, we consider a spacetime dependent $U_{ij} \equiv \alpha^a(x) t^a_{ij}$, where $\alpha^a(x)$ is the spacetime dependent gauge parameter + t^a are a basis of matrices that capture the U_{ij} rotation.

The transformation $\psi(x) \to V(x)\psi(x)$ requires $V(x)$ to be unitary + linear (or antiunitary + antilinear, i.e. like T reversal).
Assume $\mathcal{L}$ is invariant under a set of infinitesimal transformations

$$\delta \psi_\ell(x) = i\alpha^a(x)\, t^a_{\ell m}\, \psi_m(x)$$

In order to be close to identity, then any continuous symmetry can be represented by

$$V(x) = \mathbb{1} + i\alpha^a(x) t^a + \mathcal{O}(\alpha^2)$$

with $\alpha^a(x)$ real, infinitesimal, t^a are Hermitian + linear.

The comparator generalizes from

$$U(y,x) \to e^{+i\alpha(y)}\, U(y,x)\, e^{-i\alpha(x)}$$

to $$U(y,x) \to V(y)\, U(y,x)\, V^\dagger(x)$$

Then, $U(x+\epsilon n, x) = 1 + ig\,\epsilon n^\mu A^a_\mu t^a + \mathcal{O}(\epsilon^2)$

and

$$n^\mu D_\mu \psi \equiv \lim_{\epsilon\to 0} \frac{1}{\epsilon}\left[\psi(x+\epsilon n) - (1 + ig\epsilon n^\mu A^a_\mu t^a)\psi(x)\right]$$
$$= n^\mu \partial_\mu \psi - n^\mu\, ig\, A^a_\mu t^a \psi$$

And $D_\mu \psi = (\partial_\mu - ig A^a_\mu t^a)\psi$

Again, comparator puts restrictions on t^a.

$$U(x+\epsilon n, x) \to V(x+\epsilon n)\, U(x+\epsilon n, x)\, V^\dagger(x)$$

$$\Rightarrow \quad 1 + ig\epsilon n^\mu A^a_\mu t^a \to (1 + i\alpha^b(x+\epsilon n) t^b)(1 + ig\epsilon n^\mu A^a_\mu t^a)(1 - i\alpha^c(x) t^c)$$

At $\mathcal{O}(\epsilon)$, match LHS + RHS:

$$ig\epsilon n^\mu A_\mu^a t^a \to i\alpha^b(x+\epsilon n)t^b - i\alpha^c(x)t^c$$
$$+ ig\epsilon n^\mu A_\mu^a t^a - \alpha^b(x+\epsilon n)t^b g\epsilon n^\mu A_\mu^a t^a$$
$$+ g\epsilon n^\mu A_\mu^a t^a \alpha^c(x)t^c + \mathcal{O}(\alpha^2)$$
$$= ig\epsilon n^\mu A_\mu^a t^a + i(\alpha^a(x+\epsilon n) - \alpha^a(x))t^a$$
$$- g\epsilon n^\mu A_\mu^a (\alpha^b(x+\epsilon n)t^b t^a - \alpha^b(x) t^a t^b)$$

Take $\epsilon \to 0$.

$$ig A_\mu^a t^a \to ig A_\mu^a t^a + i\partial_\mu \alpha^a(x) t^a$$
$$- g A_\mu^b \alpha^c(x)(t^c t^b - t^b t^c)$$

Set $[t^b, t^c] = i f^{abc} t^a$, totally antisymmetric.

$$= ig A_\mu^a t^a + i\partial_\mu \alpha^a(x) t^a + ig A_\mu^b \alpha^c f^{abc} t^a$$

$$A_\mu^a \to A_\mu^a + \frac{1}{g}\partial_\mu \alpha^a + f^{abc} A_\mu^b \alpha^c$$

Again calculate

$$[D_\mu, D_\nu]\psi.$$

$$D_\nu \psi = \partial_\nu \psi - ig A_\nu^a t^a \psi$$

$$D_\mu D_\nu \psi = \partial_\mu\partial_\nu \psi - ig\,\partial_\mu(A_\nu^a)t^a\psi - ig A_\nu^a t^a(\partial_\mu \psi)$$
$$- ig A_\mu^b t^b(\partial_\nu \psi) - ig A_\mu^b t^b(-ig A_\nu^a t^a)\psi$$

Then $[D_\mu, D_\nu]\psi$

$$= -ig\,\partial_\mu(A_\nu^a)t^a\psi \cancel{- ig A_\nu^a t^a(\partial_\mu\psi)}$$
$$\cancel{- ig A_\mu^b t^b(\partial_\nu\psi)} - ig A_\mu^b t^b(-ig A_\nu^a t^a)\psi$$
$$+ ig\,\partial_\nu(A_\mu^a)t^a\psi \cancel{+ ig A_\mu^a t^a(\partial_\nu\psi)}$$
$$\cancel{+ ig A_\nu^b t^b(\partial_\mu\psi)} + g^2 A_\nu^b t^b A_\mu^a t^a \psi$$
$$= -ig(\partial_\mu A_\nu^a - \partial_\nu A_\mu^a)t^a\psi - g^2 A_\mu^b A_\nu^a t^b t^a \psi$$
$$+ g^2 A_\nu^b A_\mu^a t^b t^a \psi$$
$$= -ig(\partial_\mu A_\nu^a - \partial_\nu A_\mu^a) - g^2 A_\mu^a A_\nu^b (t^a t^b - t^b t^a)\psi$$

Again, $[t^a, t^b] = i f^{abc} t^c$

$$= -ig(\partial_\mu A_\nu^a - \partial_\nu A_\mu^a) - ig^2 A_\mu^a A_\nu^b f^{abc} t^c \psi$$
$$= -ig(\partial_\mu A_\nu^a - \partial_\nu A_\mu^a) - ig^2 A_\mu^b A_\nu^c f^{bca} t^a \psi$$

So $$[D_\mu, D_\nu]\psi = -ig\,(\partial_\mu A_\nu^a - \partial_\nu A_\mu^a + g f^{abc} A_\mu^b A_\nu^c) t^a \psi$$
$$= -ig\, F_{\mu\nu}^a t^a \psi$$

with $F_{\mu\nu}^a \equiv \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + g f^{abc} A_\mu^b A_\nu^c$.

Note that $F_{\mu\nu}^a$ is no longer gauge invariant by itself.

Exercise: $F_{\mu\nu}^a \to F_{\mu\nu}^a - f^{abc} \alpha^b F_{\mu\nu}^c$

Can read 15.3 in P+S for Wilson loop discussion: gives geometric interpretation of gauge variance using Stoke's theorem.

Remark: $V(x) = \mathbb{1} + i\alpha^a(x) t^a + \mathcal{O}(\alpha^2)$ is the mathematical property of <u>Lie groups</u>, which are groups with a continuous property. Will be the focus of the next lecture.