08.128.809 Theoretische Elementarteilchenphysik
Quantum Field Theory II

Homework set 3

Due June 3, 2020, by 14:00
Please note how long it took you to solve each problem!
Monday, June 1, 2020 is a holiday. We will have the discussion session for HW 3 on Monday, June 8.

2-3, 30 pts. (repeat) Using the functional method, derive the Feynman rules in momentum space for the following interaction vertices:

A. $\mathcal{L} = \lambda\phi^4$

B. $\mathcal{L} = g^2\phi^*\phi A_\mu A^\mu$. Note that for the purposes of functional methods, vector fields are simply collections of scalar fields indexed by an extra Lorentz index.

C. $\mathcal{L} = gf_{abc}\partial_\mu A_\nu^a A^{\mu,\ b} A^{\nu,\ c}$. For convenience, define all momenta to flow into the vertex, and assume f_{abc} is totally antisymmetric under interchange of any two [group space] indices.

D, Extra credit, 10 pts. $\mathcal{L} = \frac{1}{4}g^2(f^{eab}A_\mu^a A_\nu^b)(f^{ecd}A^{\mu,\ c}A^{\nu,\ d})$. Again, assume f_{abc} is totally antisymmetric under interchange of any two [group space] indices.

3-1, 25 pts. Using the functional method, derive the Feynman rules in momentum space for the following interaction vertices:

A. $\mathcal{L} = g\bar{\psi}\gamma_\mu A^\mu\psi$

B. $\mathcal{L} = y\phi\bar{\psi}\psi +$ h.c.

3-2, 75 pts. The concept of dimensional transmutation is one of the most important and involved topics in quantum field theory. We will mainly focus on the question of how a dimensionful scale arises in a physical theory when the fundamental bare Lagrangian only contains dimensionless couplings.

A, 20 pts. Read “Dimensional Analysis in Field Theory,” P. M. Stevenson. A PDF of the manuscript is available on Reader. In your own words, write a paragraph or two explaining how a dimensional scale can arise when the Lagrangian has no dimensionful couplings.

B, 25 pts. Read “Radiative Corrections as the Origin of Spontaneous Symmetry Breaking,” by S. Coleman and E. Weinberg. A PDF of the manuscript is available on Reader. In your own words, based on the discussion of massless scalar QED in section 4, write a paragraph or two explaining how a phase change in the theory can arise by a perturbative calculation.

C, 30 pts. The full calculation is done in sections II and III of the Coleman and Weinberg paper and as an exercise on pages 469-470 of Peskin and Schroeder. We will not reproduce the full calculation but instead check the most important consequences. We start with the scalar QED Lagrangian:

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + (D_\mu\phi)^\dagger D^\mu\phi - m^2\phi^\dagger\phi - \frac{\lambda}{6}\left(\phi^\dagger\phi\right)^2 , \tag{1}$$

with $D_\mu = \partial_\mu + ieA_\mu$.

i, 15 pts. Expand the Lagrangian assuming the scalar field ϕ obtains a vacuum expectation value (VEV) $\langle\phi\rangle = \phi_0$. The remaining scalar and pseudoscalar degrees of freedom of the complex scalar field ϕ can be parametrized as σ and π. This result is encoded by plugging in

$$\phi = \phi_0 + \frac{1}{\sqrt{2}}(\sigma + i\pi) \tag{2}$$

into the Lagrangian. Setting $m^2 = -\mu^2 < 0$, solve for ϕ_0 in terms of μ and λ. Identify the mass term for the A_μ vector. This is the first example of the Higgs mechanism, where the nonzero VEV of the scalar field ϕ leads to a spontaneous breaking of the QED gauge symmetry and a mass for the photon.

ii, 15 pts. Given the form of the potential in equation 4.5 of Coleman and Weinberg,

$$V = \frac{\lambda}{4}\phi_c^4 + \left(\frac{5\lambda^2}{1152\pi^2} + \frac{3e^4}{64\pi^2}\right)\phi_c^4\left(\log\frac{\phi_c^2}{M^2} - \frac{25}{6}\right) , \tag{3}$$

calculate the VEV of ϕ_c by minimizing V. Note that the potential (which includes the relevant tree-level terms from the Lagrangian in equation 4.1) does not have a tree-level mass for ϕ_c. To distinguish the behavior between when $\langle\phi_c\rangle = 0$ and $\langle\phi_c\rangle \neq 0$, specify the necessary condition on the relative magnitude of λ and e.

iii, Extra credit, 10 pts. Taking the β-functions for e and λ from Peskin and Schroeder,

$$\beta_e = \frac{e^3}{48\pi^2} , \quad \beta_\lambda = \frac{1}{24\pi^2}\left(5\lambda^2 - 18e^2\lambda + 54e^4\right) , \tag{4}$$

sketch the renormalization group flows in the (λ, e^2) plane. You should find that every trajectory passes through the condition from part ii that marks when ϕ_c acquires a vev.