Office hours, June 19, 2020

Friday, June 19, 2020 9:05 AM

SU(2)

2 = fundamental rep.

Think spin $|\uparrow\rangle$ & spin $|\downarrow\rangle$

Any SU(2) matrix can be written as lin. comb of $\sigma^i = \left\{ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -i \\ +i & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right\}$

$$\tau^i \equiv \frac{\sigma^i}{2}$$

τ^i are generators of SU(2).

Constructing invariants,

$$\bar{2} \otimes 2 = 1 \oplus 3$$

$s = \frac{1}{2}$ $\uparrow$ is $\hat{z}$-projection of $|{}^{\uparrow}_{\downarrow}\rangle$

$$|s=\tfrac{1}{2}\rangle \otimes |s=\tfrac{1}{2}\rangle = |s=0\rangle \oplus |s=1\rangle$$

$$\underbrace{\begin{matrix} |\uparrow\uparrow\rangle \\ \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle) \\ |\downarrow\downarrow\rangle \end{matrix}}_{s=1} \qquad \overbrace{\frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)}^{s=0}$$

Same algebra as SU(2) where spin angular momentum acts exactly like fundamental reps.

" $|2\rangle \times |\bar{2}\rangle = |1\rangle \oplus |3\rangle$ "

"2" = v_i = fundamental

"$\bar{2}$" $\bar{v}_j$ = antifundamental

$i, j = 1, 2$

singlet $\bar{v}_j \delta_{ji} v_i = \bar{v}_i v_i$ 2x2 or NxN

SU(2) : $v_i \to U_{ai} v_i$, $\bar{v}_j \to \bar{v}_j (U_{aj})^\dagger$

$SU(2): v_i \to U_{ai} v_i, \quad \bar{v}_j \to \bar{v}_j (U_{aj})^{\dagger}$

$$\boxed{\bar{v}_i v_i} \Rightarrow \bar{v}_i (U_{ai})^{\dagger} U_{ai} v_i$$

$$\boxed{= \bar{v}_i v_i}$$

Adjoint rep.

$$\bar{v}_j \epsilon^{ji} v_i \to \bar{v}_b \underbrace{(U_{jb})^{\dagger} \epsilon^{ji}}\, U_{ia} v_a$$

not invariant $\Big[(U_{jb})^{\dagger} U_{ia} \quad \bar{v}_b \epsilon^{ji} v_a$ → show that $\bar{v}_j \epsilon^{ji} v_i \to \epsilon^{abc} \overset{?}{\ldots} (\bar{v}_j \epsilon^{ji} v_i)$

$[\tau^i, \tau^j] = i\epsilon^{ijk} \tau^k$

ϵ^{ijk} are structure constants of $SU(2)$

by def.

$$\left([t^a, t^b] = i f^{abc} t^c \right)$$

$N^2 - 1 \to \boxed{\bar{v}_b (U_{jb})^{\dagger} \epsilon^{ji} U_{ia} v_a}$

If $j = i$, $\to 0$

dimension is 1

singlet $\to \bar{v}_b (U_{jb})^{\dagger} \delta^{ji} U_{ia} v_a$

$N \times N$ tensor

Contract $\quad \bar{v}_b v_a$

$\qquad\qquad\uparrow \;\; \uparrow$

$\qquad\qquad N \;\; N$

N^2 object

-1 singlet

$\bar{v}_b \epsilon^{ba} v_a \quad N^2 - 1$ object

↳ adjoint rep. by dimensions

ψ transforms in rep. r.

$\mathcal{L} \supset \bar{\psi} i \not{D} \psi$

$D_\mu = \partial_\mu - i g' Q \gamma^\mu$

$\qquad - i g\, t_r^a \gamma^\mu$

$-ig\, t_r^a \gamma^\mu$

τ^i are rep. t_r^a for $r = 2, \bar{2}$.

$(t^b)_{ac} = i f^{abc}$ are rep. for adj.

adj. is an $N \times N$ matrix, traceless

Acting $N \times N$ matrix by $(t^b)_{ac}$ gives another $N \times N$ matrix

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Gluons transform as adjoint.

Representation theory.

HW problem: Global symmetry?

Fields that carry extra index.

↳ $(e, \mu, \tau) = \boxed{e_i}$, $i = 1, \ldots 3$

Column vector valued field

IOW, field value is column vector

matrix-valued fields with 2 flavor indices

Scalar ϕ_{ij}

Masses: $(M^2)_{ijkl}\, \phi_{ij}\, \phi_{kl}$

When you make global symmetries gauged, still think in terms of column vector valued fermion + scalar fields

matrix valued vector fields.

$A_\mu = \underbrace{A_\mu^a\, t^a}_{\substack{\text{matrix} \\ \text{value for} \\ \text{given } x.}}$ ← matrices have restrictions to ensure the field has a "minimal" level of being matrix valued.

This is motivation for studying group theory.

Start with set of elements + obtain some unknown mathematical structure.

Identity: e element.

Mathematical structure.

Identity: e element.

But not yet interesting.

Another element: a.

Need introduce a composition operation "how do two elements interact"

Every 2-element group is $\mathbb{Z}_2$

$\otimes$	e	a
e	e	a
a	a	e

3 elements:

Keep closure.

Inversion. $\mathbb{Z}_3$ distinct from $\mathbb{Z}_2 + e$.

Most interesting aspect:

Introduce a notion of infinitesimal group element.

Two group elements can be infinitesimally close to each other.

Infinitesimal nature is carried by parameter α^a but group structure is evident already in the "basis" expansion.

$SU(N)$ = $N \times N$ matrices unitary.

$$SU(2) \ni \begin{pmatrix} 1 & 0.98 \\ -0.72 + 0.3i & 0.26 + 0.7i \end{pmatrix} = A$$

$$SU(2) \ni \begin{pmatrix} 1 & 0.980001 \\ -0.720001 + 0.300001i & 0.26 + 0.70001i \end{pmatrix} = B$$

Close if $A = \underset{\substack{\uparrow \\ \text{scaling}}}{\alpha^a} \underbrace{t^a}_{\substack{\text{basis} \\ \text{matrices}}}$ & $B = \underset{\substack{\uparrow \\ \text{infinitesimal}}}{(\alpha + \epsilon)^a} t^a$

scaling basis matrices infinitesimal

Group structure is manifest in how t^a act with each other.

For example, $\alpha \neq I$, then $\epsilon \neq 0$.

Each t^a is a unique "direction" in the $N \times N$ matrix space.

If all $t^a = \left\{ \begin{pmatrix} (2\times 2) & 0 \\ 0 & (3\times 3) \end{pmatrix} \right\}$

not minimal.

Conversely, subset of t^a is insufficient $\Rightarrow$ need closure.