Lecture 9, May 25, 2020

Monday, May 25, 2020 9:11 AM

Last time: Began path integrals + functional methods.

Today: Examples with functional methods for scalars and fermionic path integrals.

Announcements:

Holiday on Monday, June 1.

HW3 due on June 3. (Typo in eqn. 1 ⇒ $-\frac{\lambda}{6}(\phi^\dagger\phi) \to -\frac{\lambda}{6}(\phi^\dagger\phi)^2$)

Generating functional

$$Z[J] \equiv \int D\phi \, \exp\left(i \int d^4x \left[\mathcal{L} + J(x)\phi(x) \right] \right)$$

$$Z[J_1, J_2] = \int D\phi \, D\phi^* \exp\left[i \int d^4x \, \mathcal{L} + J_1\phi + J_2\phi^* \right]$$

n-pt. correlation fcn.

$$\langle \Omega | T\phi(x_1) \dots \phi(x_n) | \Omega \rangle = \frac{1}{Z_0} \left(\frac{-i\delta}{\delta J(x_1)} \right) \left(\frac{-i\delta}{\delta J(x_2)} \right) \cdots \left(\frac{-i\delta}{\delta J(x_n)} \right) Z[J] \Big|_{J=0}$$

$$Z_0 \equiv Z[J=0]$$

Intuitively, have $\mathcal{L}$ (can be very complicated), and throw in sources + see response.

Considering $J \neq 0$ + calculating convolution from the path integral, we get amplitude for all possible n-pt. correlation fcns + need select the order which we need + reset $J=0$.

Example: (Cheng + Li, Appendix B)

Consider scalar field theory.

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_{int}$$

$$\mathcal{L}_0 = \frac{1}{2}(\partial_\mu\phi)^2 - \frac{1}{2}\mu^2\phi^2$$

$$\mathcal{L}_{int} = -\frac{\lambda}{4!}\phi^4$$

$$W[J] = \int [d\phi] \exp\left[i \int \left[\mathcal{L}_0 + \mathcal{L}_{int} + J(x)\phi(x) \right] d^4x \right]$$

$$= \exp\left[i \int d^4x \, \mathcal{L}_I\left(\frac{-i\partial}{\partial J} \right) \right] W_0[J]$$

↳ free theory

you replace ϕ by $\left(\frac{-i\delta}{\delta J} \right)$

$W_0[J] = W[J]$ without $\mathcal{L}_{int}$.

Solve free theory:

$$W_0[J] = \int [d\phi] \exp\left[i\int \left(\tfrac{1}{2}(\partial_\mu\phi)^2 - \tfrac{1}{2}\mu^2\phi^2 + J\phi\right) d^4x \right]$$

$$= \int [d\phi] \exp\left[i\int \left(\tfrac{1}{2}\phi P \phi + J\phi\right) d^4x \right]$$

where $P(x) \equiv -(\partial^2 + \mu^2 - i\epsilon)$

P is Hermitian operator (Klein-Gordon operator)

+ inserted $\exp\left[-\frac{1}{2}\int \epsilon \phi^2 d^4x\right]$ to make the path integral convergent.

Let us assume ϕ_c is the classical field solution in the presence of external source J

$$P(x)\,\phi_c(x) = -J(x)$$

Using Green's fcn. techniques, we can invert this equation by defining

$$P(x)\,\Delta_F(x-y) = \delta^{(4)}(x-y)$$

$$\phi_c(x) = -\int \Delta_F(x-y) J(y)\, d^4y$$

Solve for $\Delta_F(x) = \int \frac{d^4k}{(2\pi)^4} e^{-ik\cdot x} \Delta_F(k)$

$$\Delta_F(k) = \frac{1}{k^2 - \mu^2 + i\epsilon}$$

Now, consider fluctuations around classical field.

Change integration variables from ϕ to ϕ'

$$\phi(x) = \phi_c(x) + \phi'(x).$$

Then, $W_0[J] = N \exp\left[-i\int d^4x\, d^4y \left[\tfrac{1}{2} J(x)\Delta_F(x-y)J(y)\right]\right]$

by the usual case of Gaussian integration.

$N \equiv \int [d\phi'] \exp\left[i\int \tfrac{1}{2}\phi'(x) P(x) \phi'(x)\, d^4x\right]$ is ind. of J, will cancel out

$$W[J] = N \exp\left[i\int d^4x\, \mathcal{L}_i\left(\frac{-i\,\delta}{\delta J}\right)\right] \cdot \exp\left[\frac{-i}{2}\int d^4x_1 d^4x_2\, J(x_1)\Delta_F(x_1-x_2)J(x_2)\right]$$

Recall rule for differentiating functionals

$$\frac{\delta J(y)}{\delta J(x)} = \frac{\delta}{\delta J(x)} \int \delta^4(y-x) J(x)\, d^4x = \delta^{(4)}(x-y)$$

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then we produce Wick's theorem in $W[J]$

Connected Green's fcns. are obtained by differentiating the $\ln W[J]$ functional

$$G^{(n)}(x_1 \ldots x_n) = \langle 0 | T(\phi(x_1) \ldots \phi(x_n)) | 0 \rangle_{conn.}$$

$$= i^n \frac{\delta^n \ln W[J]}{\delta J(x_1) \ldots \delta J(x_n)} \Bigg|_{J=0}$$

Simplest example:

two pt. fcn.

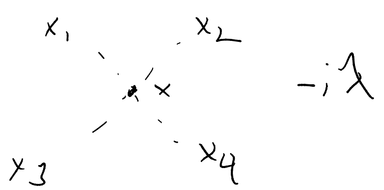
$$\langle 0 | T(\phi(x)\phi(y)) | 0 \rangle_{free} = i^2 \frac{\delta^2 \ln W_0[J]}{\delta J(x)\, \delta J(y)} \Bigg|_{J=0}$$

$$= i\Delta_F(x-y)$$

4 pt. fcn.

$$G^{(4)}(x_1 \ldots x_4) = -i\lambda \int d^4x$$

$$\cdot\, i\Delta_F(x_1-x)\, i\Delta_F(x_2-x)\, i\Delta_F(x_3-x)\, i\Delta_F(x_4-x)$$



As an exercise, can perform the differentiation + get all terms with various powers of J. But we discard extra powers of J by setting $J=0$ at end.

Second example:

Scalar QED: derive Feynman rule for complex scalar interacting with photon.

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu}^2 + (D_\mu\phi)^* (D^\mu\phi) - m^2 \phi^*\phi$$

$$D_\mu = \partial_\mu + ieA_\mu$$

$\phi\phi^* A_\mu$ vertex

$$\mathcal{L} = \mathcal{L}_0 + (-ieA_\mu\phi^*)\partial^\mu\phi + (\partial_\mu\phi^*)(ieA^\mu\phi)$$

Taylor expansion of interaction term gives

$$\exp\left[i\int d^4x \left(-ieA_\mu\phi^*(\partial^\mu\phi) + (\partial_\mu\phi^*)ieA^\mu\phi\right)\right.$$

$$\simeq 1 + i\int d^4x \left(-ieA_\mu\phi^*(\partial^\mu\phi) + (\partial_\mu\phi^*)ieA^\mu\phi\right)$$

$$= 1 + \int d^4x\, eA_\mu\phi^*\partial^\mu\left(\int\frac{d^4k_1}{(2\pi)^4}\cdot e^{-ik_1\cdot x}\phi(k_1)\right)$$

$$-\int d^4x\, eA_\mu\phi\,\partial_\mu\left(\int\frac{d^4k_2}{(2\pi)^4}\cdot e^{+ik_2\cdot x}\phi(k_2)\right)$$

$$= 1 + \int d^4x\, e\cancel{A_\mu}\cancel{\phi^*}(-ik_1^\mu)\cancel{\phi(x)}$$

$$-\int d^4x\, e\cancel{A_\mu}(ik_2^\mu))\cancel{\phi^*(x)}\cancel{\phi}$$

$$\langle\Omega|T(A_\mu\phi\phi^*)|\Omega\rangle = \exp\left(i\int d^4x\, A_\mu\phi\partial^\mu\phi^*\right)$$

In the functional integral, this gets integrated against $\exp\left[\int_{\uparrow} i\int d^4x\,\mathcal{L}_0\right]$

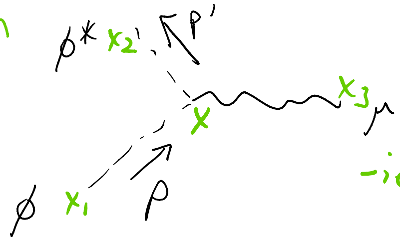
path integral

$$\int[d\phi][d\phi^*][dA_\mu]\left[\mathcal{L}_0 + \underline{J_1\phi} + \underline{J_2^\dagger\phi^*} + \underline{J_3^\mu A_\mu}\right)$$

↳ free Lagrangian

↳ extracts free field propagators for each field as external particles, which we end up removing for connected Green's fcn.

The integration gives δ-fcns. for any field in the interaction term, leaving only the coefficient.



$-ie(p+p')^\mu$ from above.

$$-ie(p+p')^\mu\underbrace{\left(i\Delta_F(x_1-x)\, i\Delta_F^*(x_2-x)\cdot i\Delta_A(x_3-x)\right)}_{\text{Imposes momentum conservation on vtx.}}$$

Alternatively, take $\mathcal{L}_{int}$ term + add i for perturbative expansion.

$$i(-ieA_\mu\phi^*)(\partial^\mu\phi) + i(\partial_\mu\phi^*)ieA^\mu\phi$$

$$\star = eA_\mu\phi^*\partial^\mu\phi - e(\partial_\mu\phi^*)A^\mu\phi$$

For each term, unique way to contract $A_\alpha\phi\phi^*$

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(in other words, fcnal derivative will integrate to δ-fcns)

Adopt momentum convention

$\partial^\mu \phi \to -ip^\mu$ incoming

$\partial^\mu \phi^* \to ip'^\mu$ outgoing

fcnal derivatives on $\star$: $\left(eA_\mu \phi^* \partial^\mu \phi - \partial_\mu \phi^* e A^\mu \phi\right) \cdot \frac{\delta}{\delta A_\alpha} \frac{\delta}{\delta \phi^*} \frac{\delta}{\delta \phi}$

$$= e\,\delta_{\mu\alpha}(-ip^\mu) - ip'^\mu e\,\delta_{\alpha\nu} g^{\mu\nu} = -ie(p+p')^\mu$$

$$\frac{\delta}{\delta A_\alpha} A_\mu = \delta_{\mu\alpha}$$

$$\frac{\delta}{\delta A_\alpha} A^\mu = \frac{\delta}{\delta A_\alpha} A_\nu g^{\mu\nu} = g^{\mu\nu} \frac{\delta A_\nu}{\delta A_\alpha} = g^{\mu\nu} \delta_{\alpha\nu}$$

This method of fcnal derivatives also makes it easier to count symmetry factors.

$$\frac{\delta}{\delta \phi(x_1)} \frac{\delta}{\delta \phi(x_2)} \frac{\delta}{\delta \phi(x_3)} \frac{\delta}{\delta \phi(x_4)} \cdot \left[\frac{\lambda}{4!} \cdot \phi^4(x)\right]$$

$\Rightarrow$ 4! ways of contracting the derivs. $\Rightarrow$ get λ as overall coupling.

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fermions in path integrals:

Introduce Grassmann numbers, which are anti-commuting numbers.

Rules: $\theta \eta = -\eta \theta$; $\theta^2 = 0$.

$$\int d\theta\, f(\theta) = \int d\theta\, (A + B\theta) \equiv B.$$

from invariance under shift.

$$\int d\theta \int d\eta\, \eta\, \theta = +1$$, innermost integral first.

Complex Grassmann: $\theta = \frac{\theta_1 + i\theta_2}{\sqrt{2}}$, $\theta^* = \frac{\theta_1 - i\theta_2}{}$

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complex conjugate reverses order

$$(\theta\eta)^* = \eta^*\theta^* = -\theta^*\eta^*$$

$$\int d\theta^* d\theta \;\theta\theta^* = 1, \quad \theta \;\&\; \theta^* \text{ are independent.}$$

Evaluate Gaussian:

$$\int d\theta^* d\theta \; e^{-\theta^* b\theta} = \int d\theta^* d\theta \,(1 - \theta^* b\theta)$$

$$= \int d\theta^* d\theta \,(1 + \theta b\theta^*)$$

$$= \int d\theta^* d\theta \;\theta b\theta^*$$

$$= b$$

Note contrast: $\int d\xi \, e^{-b\xi^2} = \sqrt{\frac{\pi}{b}}$, $\int d\xi \,\xi\, e^{-b\xi^2} = 0$

$$\int d\xi \,\xi^2 e^{-b\xi^2} = \frac{1}{2b}\sqrt{\frac{\pi}{b}}$$

Also,

$$\int d\theta^* d\theta \;\theta\theta^* \, e^{-\theta^* b\theta} = 1$$

In general, Taylor expansions of functions of Grassmann numbers truncate quickly. So $\prod_i \int d\theta_i^* d\theta_i \, f(\theta)$ select one factor of each θ_i + θ_i^*.

Also, for $\left(\prod_i \int d\theta_i^* d\theta_i\right) e^{-\theta_i^* B_{ij}\theta_j}$ for B Hermitian

$$= \left(\prod_i \int d\theta_i^* d\theta_i\right) e^{-\sum_i \theta_i^* b_i \theta_i} = \prod_i b_i = \det B.$$

Dirac field $\psi(x) = \sum \psi_i \phi_i(x)$

ϕ_i four-component spinors, ψ_i are Grassmann numbers.

Then,

$$\langle 0 | T\psi(x_1)\bar\psi(x_2) | 0\rangle$$

$$= \int \frac{\mathcal{D}\bar{\Psi}\,\mathcal{D}\Psi \, \exp\left[i\int d^4x\, \bar{\Psi}(i\partial\!\!\!/-m)\Psi\right]\Psi(x_1)\bar{\Psi}(x_2)}{\int \mathcal{D}\bar{\Psi}\,\mathcal{D}\Psi \, \exp\left(i\int d^4x\, \bar{\Psi}(i\partial\!\!\!/-m)\Psi\right)}$$

Check: $\langle 0|\, T\, \Psi(x_1)\,\bar{\Psi}(x_2)\,|0\rangle = S_F(x_1-x_2) = \int \frac{d^4k}{(2\pi)^4} \frac{i e^{-ik\cdot(x_1-x_2)}}{k\!\!\!/ - m + i\epsilon}$

Generating functional:

$$Z[\bar{\eta},\eta] = \int \mathcal{D}\bar{\Psi}\,\mathcal{D}\Psi \, \exp\left[i\int d^4x\left(\bar{\Psi}(i\partial\!\!\!/-m)\Psi + \bar{\eta}\Psi + \bar{\Psi}\eta\right)\right]$$

$$= Z_0 \exp\left[-\int d^4x\, d^4y\, \bar{\eta}(x)\, S_F(x-y)\, \eta(y)\right]$$

set external sources to 0.

exactly analogous to solution in scalar free theory.

Define sign convention

$$\frac{d}{d\eta}\theta\eta = -\frac{d}{d\eta}\eta\theta = -\theta$$

Get $\langle 0|\, T\, \Psi(x_1)\bar{\Psi}(x_2)\,|0\rangle = Z_0^{-1}\left(\frac{-i\delta}{\delta\bar{\eta}(x_1)}\right)\left(\frac{+i\delta}{\delta\eta(x_2)}\right) Z[\bar{\eta},\eta]\Big|_{\bar{\eta},\eta=0}$

Example: QED vertex

$$\mathcal{L} = \bar{\Psi}(iD\!\!\!\!/ - m)\Psi - \frac{1}{4}F_{\mu\nu}^2$$

$$= \bar{\Psi}(i\partial\!\!\!/ - m)\Psi - e\bar{\Psi}\gamma_\mu\Psi A_\mu - \frac{1}{4}(F_{\mu\nu})^2$$

$$\exp\left[i\int d^4x\,\mathcal{L}\right] = \exp\left[i\int \mathcal{L}_0\, d^4x\right]\left[1 - ie\int \cancel{d^4x}\,\cancel{\bar{\Psi}}\,\gamma^\mu\,\cancel{\Psi}\,\cancel{A_\mu} + \dots\right]$$

$\Rightarrow$ Fcnal derivatives give $-ie\gamma^\mu$