Lecture 9.
QFT II.
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Last time: Introduced path integrals and functional methods
Today: More functional integrals, examples of $\lambda\phi^4$ and scalar QED.
Fermionic path integrals

Following Cheng + Li, Appendix B.
Also see Schwartz, 14.2 + 14.3.

First, solve free $\mathcal{L}_0 = \frac{1}{2}(\partial_\mu\phi)^2 - \frac{1}{2}\mu^2\phi^2$

$$\mathcal{L}_{int} = -\frac{\lambda}{4!}\phi^4$$

$$Z[J] = \int D\phi \exp\left[i\int d^4x \left(\mathcal{L}_0 + \mathcal{L}_{int} + J(x)\phi(x)\right) d^4x\right.$$

or, using Cheng + Li notation,

$$W[J] = \int [d\phi] \exp\left\{i\int [\mathcal{L}_0 + \mathcal{L}_{int} + J(x)\phi(x)]\, d^4x\right.$$

$$= \exp\left[i\int d^4x\, \mathcal{L}_I\left(\frac{-i\partial}{\partial J}\right) W_0[J]\right.$$

where we replace ϕ by $\frac{-i\partial}{\partial J}$ fcnal deriv.

$$W_0[J] = \int [d\phi] \exp\left[i\int d^4x\, (\mathcal{L}_0(x) + J(x)\phi(x))\right]$$

is the free theory generating fcnal.

Solve free theory:

$$W_0[J] = \int [d\phi] \exp\left[i\int \left(\frac{1}{2}(\partial_\mu\phi)^2 - \frac{1}{2}\mu^2\phi^2 + J(x)\phi(x)\right) d^4x\right]$$

$$= \int [d\phi] \exp\left[i\int \left(\frac{1}{2}\phi P(x)\phi(x) + J(x)\phi(x)\right) d^4x\right]$$

for operator $P(x) = -(\partial^2 + \mu^2 - i\epsilon)$, where we introduce term $\exp\left(-\frac{1}{2}\left(\int \epsilon\phi^2 d^4x\right)\right)$, $\epsilon > 0$, for convergence of path integral.

Let $\phi_c(x)$ be soln. of P, classical free-field eqn., in presence of ext. source J.

$$P\phi_c = -J$$

Then, using Green's fcn, we can invert this eqn.

$$P(x)\,\Delta_F(x-y) = \delta^4(x-y)$$

$$\phi_c(x) = -\int \Delta_F(x-y)\,J(y)\,d^4y \quad \text{, convolution of } \Delta_F + J$$

Can determine $\Delta_F = \int \frac{d^4k}{(2\pi)^4} e^{-ik\cdot x} \Delta_F(k)$, Fourier transform

$$\Delta_F(k) = \frac{1}{k^2-\mu^2+i\epsilon}$$

We now change variables:

$$\phi(x) = \phi_c(x) + \phi'(x)$$

So $W_0[J] = \int [d\phi] \exp\left(i\int(\tfrac{1}{2}\phi P\phi + J\phi)\,d^4x\right)$

Jacobian is 1.

$$= \int [d\phi'] \exp\left(i\int \left(\tfrac{1}{2}(\phi_c+\phi')P(\phi_c+\phi') + J(\phi_c+\phi')\right)d^4x\right)$$

$$= \int [d\phi'] \exp\Big(i\int \tfrac{1}{2}(\phi_c P\phi_c + \phi_c P\phi' + \phi' P\phi_c + \phi' P\phi'$$
$$+ J\phi_c + J\phi')\,d^4x$$

Use $P\phi_c = -J, \Rightarrow J\phi_c = -\phi_c P\phi_c$

P is Hermitian

$$= \int [d\phi'] \exp\left(i\int \tfrac{1}{2}(\phi_c P\phi_c - 2\phi_c P\phi_c + \phi' P\phi')\,d^4x\right)$$

$$= \int [d\phi'] \exp\left(i\int \tfrac{1}{2}(\phi' P\phi' - \phi_c P\phi_c)\,d^4x\right)$$

$$W_0[J] = N \exp\left(\tfrac{-i}{2}\int \phi_c P\phi_c\,d^4x\right)$$

$$N \equiv \int [d\phi'] \exp\left(\tfrac{i}{2}\int \phi' P\phi'\,d^4x\right)$$

Use $\phi_c = -\int \Delta_F(x-y)\,J(y)\,d^4y$, $P(x)\,\phi_c(x) = -J(x)$

$$W_0[J] = N \exp\left(\tfrac{-i}{2}\int \left(-\int \Delta_F(x-y)\,J(y)\,d^4y\right)\cdot(-J(x))\,d^4x\right)$$

$$= N \exp\left(\tfrac{-i}{2}\iint J(x)\,\Delta_F(x-y)\,J(y)\,d^4x\,d^4y\right)$$

Now, $W[J] = \exp\left[i\int d^4x\, \mathcal{L}_I\left(\frac{-i\,\partial}{\partial J}\right)\right] W_0[J]$

Consider $\mathcal{L}_I = \frac{-\lambda}{4!}\phi^4$

$$W[J] = \exp\left[i\int d^4z \left(\frac{-\lambda}{4!}\right)\left(\left(\frac{-i\partial}{\partial J}\right)^4\right)\right]$$

$$\cdot N \exp\left(\frac{-i}{2}\iint J(x)\,\Delta(x-y)\,J(y)\,d^4x\,d^4y\right)$$

$$\simeq \left(1 - \frac{i\lambda}{4!}\int d^4z \left(\frac{-i\partial}{\partial J}\right)^4 + \dots\right) N\exp\left(\frac{-i}{2}\iint d^4x\,d^4y\; J(x)\Delta_F J(y)\right)$$

Consider first order in $W[J]$: Proceed step by step

$$W[J] = -i\frac{\lambda}{4!} N\int d^4z \left(\frac{-i\partial}{\partial J}\right)^4 \exp\left[\frac{-i}{2}\iint d^4x\,d^4y\; J(x)\Delta_F(x-y)J(y)\right]$$

$$= -\frac{i\lambda}{4!} N\int d^4z \left(\frac{-i\partial}{\partial J}\right)^3\left(\frac{-1}{2}\int d^4y\,\Delta_F(z-y)J(y) - \frac{1}{2}\int d^4x\,J(x)\Delta_F(x-z)\right)$$

$$\exp\left[\frac{-i}{2}\iint d^4x\,d^4y\; J(x)\Delta_F(x-y)J(y)\right]$$

$$= -\frac{i\lambda}{4!} N\int d^4z \left(\frac{-i\partial}{\partial J}\right)^3\left(-\int d^4x\,J(x)\Delta_F(x-z)\exp[\dots]\right)$$

$$= -\frac{i\lambda}{4!} N\int d^4z \left(\frac{-i\partial}{\partial J}\right)^2\left(i\Delta_F(z-z) + \int d^4x\,J(x)\Delta_F(x-z)\right.$$
$$\left.\int d^4x'\,J(x')\Delta_F(x'-z)\right)\exp[\dots]$$

$$= -\frac{i\lambda}{4!} N\int d^4z \left(\frac{-i\partial}{\partial J}\right)\left(\Delta_F(z-z)\int \Delta_F(z-y)J(y)\,d^4y\right.$$
$$+ (-i)\,2\Delta_F(z-z)\int d^4x\,J(x)\Delta_F(x-z)$$
$$+ \int d^4x\,J(x)\Delta_F(x-z)\int d^4x'\,J(x')\Delta_F(x'-z)$$
$$\left.\int d^4x''\,J(x'')\Delta_F(x''-z)\right)\exp[\dots]$$

$$= -\frac{i\lambda}{4!} N\int d^4z \left(\Delta_F(z-z)\Delta_F(z-z) + \right.$$
$$\Delta_F(z-z)\int \Delta_F(z-y)J(y)\,d^4y\int d^4y'\,\Delta(z-y')J(y')$$
$$-2\Delta_F(z-z)\,\Delta_F(z-z)$$
$$-(2\Delta_F(z-z))\int d^4x\,J(x)\Delta_F(x-z)\int d^4y\,J(y)\Delta_F(y-z)$$
$$+ 3\Delta_F(z-z)\int d^4x\,J(x)\Delta_F(x-z)\int d^4y\,J(y)\Delta_F(y-z)$$
$$+ \int d^4x\,J(x)\Delta_F(x-z)\int d^4x'\,J(x')\Delta_F(x'-z)$$
$$\left.\int d^4x''\,J(x'')\Delta_F(x''-z)\int d^4x'''\,J(x''')\Delta_F(x'''-z)\right)$$
$$\exp[\dots]$$

The four-pt. correlation fcn.

$$\langle\Omega| T\,\phi(x_1)\,\phi(x_2)\,\phi(x_3)\,\phi(x_4)\,|\Omega\rangle$$
$$= \frac{1}{Z_0}\left(\frac{-i\delta}{\delta J(x_1)}\,\frac{-i\delta}{\delta J(x_2)}\,\frac{-i\delta}{\delta J(x_3)}\,\frac{-i\delta}{\delta J(x_4)}\right) Z[J]\Big|_{J=0}$$

Extract

$$\langle\Omega| T\phi^4 |\Omega\rangle = \frac{-i\lambda}{4!}\left(\prod_{i=1}\frac{-i\delta}{\delta J(x_i)}\right)\int d^4z.$$
$$\int dx\, J(x)\Delta_F(x-z)\int d^4x'\, J(x')\,\Delta_F(x'-z)$$
$$\int d^4x''\, J(x'')\Delta_F(x''-z)\int d^4x'''\, J(x''')\Delta_F(x'''-z)$$
$$\exp[\ldots]\Big|_{J=0}$$
$$= -i\lambda\int d^4z\;\Delta_F(x-z)\,\Delta_F(x'-z)\,\Delta_F(x''-z)\,\Delta_F(x'''-z)$$

Summary: fields replaced by $\left(\frac{-i\partial}{\partial J}\right)$, then extracted by $\prod_{i=1}^{N}\left(\frac{-i\delta}{\delta J(x_i)}\right)$. Get position space propagators and vertex fcn., where vtx. fcn. is the feynman rule.

Second example: Peskin + Schroeder problem 9.2, scalar QED.

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}^2 + (D_\mu\phi)^*(D_\mu\phi) - m^2\phi^*\phi$$
$$D_\mu = \partial_\mu + ieA_\mu$$

Extract $\phi\phi^* A_\mu$ vertex.

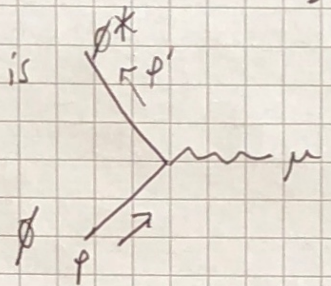
$$\mathcal{L} = \mathcal{L}_0 + (-ieA_\mu\phi^*)\,\partial^\mu\phi + (\partial_\mu\phi^*)(ieA^\mu\phi)$$

Taylor expansion of interaction term gives

$$\exp\left[i\int d^4x\,\left(-ieA_\mu\phi^*(\partial^\mu\phi)\right) + (\partial_\mu\phi^*)\,ie\,A^\mu\phi\right]$$
$$\simeq 1 + i\int d^4x\,\left(-ieA_\mu\phi^*(\partial^\mu\phi) + (\partial_\mu\phi^*)\,ieA^\mu\phi\right)$$
$$= 1 + \int d^4x\; eA_\mu\phi^*\,\partial^\mu\left(\int\frac{d^4k_1}{(2\pi)^4}\,e^{-ik_1\cdot x}\,\phi(k_1)\right)$$
$$-\int d^4x\,\left(\partial_\mu\left(\int\frac{d^4k_2}{(2\pi)^4}\,e^{ik_2\cdot x}\,\phi(k_2)\right)eA^\mu\phi\right.$$

$$= 1 + \int d^4x \; eA_\mu \phi^* (-ik_1^\mu)\, \phi(x) - \int d^4x \; eA^\mu (ik_{2\mu})\, \phi^*(x)\, \phi$$

So Feynman rule is



$$-ie(p+p')^\mu$$

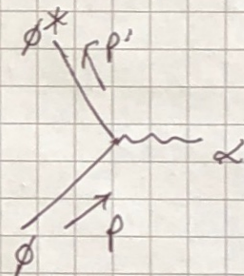
Alternatively, take $\mathcal{L}_{int}$ term and add i for perturbation exp.

$$i(-ieA_\mu \phi^*)(\partial^\mu \phi) + i(\partial_\mu \phi^*)\, ieA^\mu \phi$$
$$= eA_\mu \phi^* \partial^\mu \phi - (\partial_\mu \phi^*)\, eA^\mu \phi$$

For each term, can contract $A_\alpha \phi \phi^*$ uniquely.

Adopt momentum convention.

$$\partial^\mu \phi \to -ip^\mu \text{ incoming}$$
$$\partial^\mu \phi^* \to ip'^\mu \text{ outgoing}$$



$$\frac{\delta}{\delta A_\alpha} \frac{\delta}{\delta \phi} \frac{\delta}{\delta \phi^*} \left(eA_\mu \phi^* \partial^\mu \phi - \partial_\mu \phi^* eA^\mu \phi \right)$$

↙ Kronecker-δ-fcn.

$$\Rightarrow e\,\delta_{\mu\alpha} (-ip^\mu) - ip'^\mu e\, \delta_{\alpha\nu} g^{\mu\nu} = -ie(p+p')^\mu$$

Fermions in path integrals

Main complication is needing to introduce anticommuting numbers (Grassmann numbers).

$$\theta\eta = -\eta\theta, \text{ so } \theta^2 = 0$$

$$\int d\theta\, f(\theta) = \int d\theta\, (A + B\theta) = B,$$

from invariance under shift of integration variable.

$$\int d\theta \int d\eta\; \eta\theta = +1, \text{ innermost integral first}$$

Complex Grassmann: reverse order
↗ conjugate

$$(\theta\eta)^* = \eta^* \theta^* = -\theta^* \eta^*$$

Define $\theta = \frac{\theta_1 + i\theta_2}{\sqrt{2}}$, $\theta^* = \frac{\theta_1 - i\theta_2}{\sqrt{2}}$

$\int d\theta^* d\theta \, (\theta\theta^*) = 1$, θ + θ^* independent.

Evaluate Gaussian

$$\int d\theta^* d\theta \, (\theta\theta^*) = 1.$$

Note contrast: $\int d\xi \, e^{-b\xi^2} = \sqrt{\frac{\pi}{b}}$, $\int d\xi \, \xi e^{-b\xi^2} = 0$, $\int d\xi \, \xi^2 e^{-b\xi^2} = \frac{1}{2b}\sqrt{\frac{\pi}{b}}$

Also, $\int d\theta^* d\theta \, \theta\theta^* e^{-\theta^* b\theta} = 1$

In general, $\prod_i \int d\theta_i^* d\theta_i \, f(\theta)$ selects one factor of each θ_i + θ_i^* from $f(\theta)$.

Also, $\left(\prod_i \int d\theta_i^* d\theta_i\right) e^{-\theta_i^* B_{ij} \theta_j}$

$$= \left(\prod_i \int d\theta_i^* d\theta_i\right) e^{-\sum_i \theta_i^* b_i \theta_i} = \prod_i b_i = \det B$$

for B Hermitian.

Dirac field $\Psi(x) = \sum_i \psi_i \phi_i(x)$

ϕ_i four-component spinors, ψ_i are Grassmann numbers.

Then $\langle 0 | T\Psi(x_1)\bar\Psi(x_2) | 0 \rangle$

$$= \frac{\int D\bar\Psi D\Psi \exp[i\int d^4x \, \bar\Psi(i\not\partial - m)\Psi] \Psi(x_1)\bar\Psi(x_2)}{\int D\bar\Psi D\Psi \exp[i\int d^4x \, \bar\Psi(i\not\partial - m)\Psi]}$$

Check: $\langle 0 | T\Psi(x_1)\bar\Psi(x_2) | 0\rangle = S_F(x_1 - x_2) = \int \frac{d^4k}{(2\pi)^4} \frac{i e^{-ik\cdot(x_1 - x_2)}}{\not k - m + i\epsilon}$

Generating functional:

$$Z[\bar\eta, \eta] = \int D\bar\Psi D\Psi \exp[i\int d^4x \, [\bar\Psi(i\not\partial - m)\Psi + \bar\eta\Psi + \bar\Psi\eta]]$$

$$= Z_0 \exp[-\int d^4x \int d^4y \, \bar\eta(x) S_F(x-y) \eta(y)]$$

Z_0: set external sources to 0.

Define sign convention

$$\frac{d}{d\eta}\theta\eta = -\frac{d}{d\eta}\eta\theta = -\theta$$

Get $$\langle 0|\, T\psi(x_1)\bar{\psi}(x_2)\,|0\rangle = Z_0^{-1}\left(\frac{-i\delta}{\delta\bar{\eta}(x_1)}\right)\left(\frac{+i\delta}{\delta\eta(x_2)}\right) Z[\bar{\eta},\eta]\Big|_{\eta,\bar{\eta}=0}$$

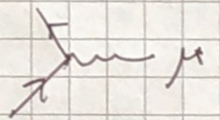
Example:

$$\mathcal{L}_{QED} = \bar{\psi}(i\not{D}-m)\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}$$
$$= \bar{\psi}(i\not{\partial}-m)\psi - e\bar{\psi}\gamma^\mu\psi A_\mu - \frac{1}{4}(F_{\mu\nu})^2$$

$$\not{D} = \not{\partial} + ie\not{A}$$

$$\exp\left[i\int\mathcal{L}\,d^4x\right] = \exp\left[i\int\mathcal{L}_0\,d^4x\right]\left(1 - ie\int d^4x\,\bar{\psi}\gamma^\mu\psi A_\mu + \dots\right)$$

Functional derivatives give $\boxed{-ie\gamma^\mu}$ for



Feynman rule.