Lecture 8, May 20, 2020

Wednesday, May 13, 2020 7:03 PM

Last time: Wrapped up renormalization

Today: Path integrals + functional methods

Announcements:

HW3 is posted on Reader: Due on June 3 @ 14:00 by email.

Discussion session for HW3 will be on June 8.

June 1 is a holiday.

Remarks: A main focus of renormalization is the running of coupling constants.

cf. P2 experiment @ Mainz to measure world's best $\sin^2\theta_W$ at very low Q^2

Another very deep result from renormalization is the generation of dimensionful parameters from dimensionless couplings.

"Dimensional transmutation" - main focus of HW3.

Puzzle: Start from $\mathcal{L}$ w/ no dimensionful couplings, but then a physical scale arises in the solution.

Example: Coleman-Weinberg; massless scalar QED

Some choices of parameters of $e, \lambda \Rightarrow$ generate a vacuum expectation value for ϕ + makes a mass for photon.

This phenomenon is known as Higgs mechanism.

Mass for photon

$U(1)$ theory:

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + |D_\mu \phi|^2$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

$$\mathcal{L}_{mass} = m^2 A_\mu A^\mu \Leftarrow \text{breaks gauge invariance.}$$

$$A_\mu \to A_\mu + \partial_\mu \alpha$$

α ← gauge parameter

Spontaneous breaking; ~ dynamical

gauge parameter

Spontaneous breaking, dynamical field

$$\phi = \underset{\substack{\uparrow \\ \text{constant}}}{\phi_0} + \tilde{\phi}$$

constant value of field ϕ for all x ≡ vacuum expectation value, $\langle \phi \rangle = \phi_0$

← gauge variation of original ϕ is now encoded in $\tilde{\phi}$.

Study effect of ϕ_0: $D_\mu \phi = \partial_\mu \phi + ie A_\mu \phi$

$\partial_\mu \phi_0 = 0$

$ie A_\mu \phi_0 \neq 0$

$$\mathcal{L} \supset -\frac{1}{4} F_{\mu\nu}F^{\mu\nu} + |\underbrace{ie A_\mu \phi_0}|^2$$

$$= e^2 \phi_0^2 A_\mu A^\mu$$

same as mass term for $A_\mu A^\mu$ for $m^2 = e^2\phi_0^2$.

Higgs mechanism:

For scalar field charged under gauge symmetry, VEV for scalar provides mass term for gauge field

$m^2 \sim g^2 v^2$.

Key point about spont. breaking. You start w/ gauge invariant Lagrangian + expand around nonzero vev. Lagrangian is still gauge invariant ⇒ gauge symmetry is generally not linearly realized but non-linearly realized.

New unit

Path integrals & functional methods.

Path integral formalism gives an alternative approach to calculating correlation fcns.

Path integrals in quantum mechanics:

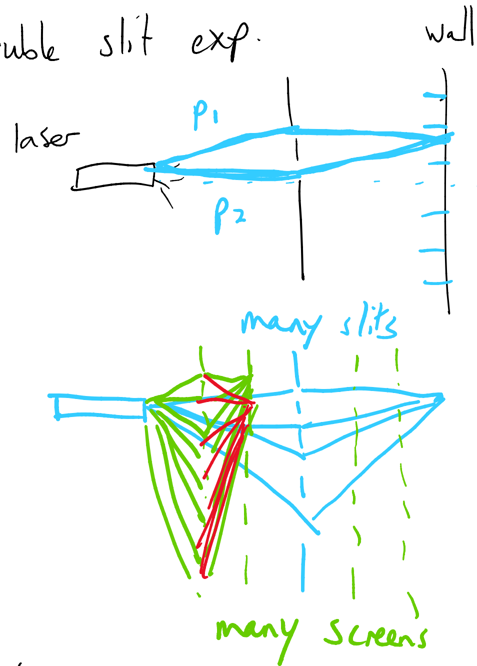
Calculating amplitudes for particle to travel from x_A to x_B in T.

$$U(x_A, x_B; T) = \langle x_b | e^{-iHT/\hbar} | x_a \rangle$$

Can formulate this as x_a going to x_b over all possible paths, with each path contributing some phase.

Geometric interpretation: Feynman thought experiment
∞ slits, ∞ screens.

Double slit exp.



amplitude h given by interference (phase difference) of two paths between slits.

Sum over all slits & screens = geometric interpretation of path integral.

$$U(x_a, x_b; T) = \int \mathcal{D}x(t)\, e^{i(\text{phase})}$$

The phase is the action, $S = \int L\, dt = \int \mathcal{L}\, d^4x$
where the classical path is obtained by method of stationary phase $(\hbar \to 0)$.

Equivalence between

$$\langle x_b | e^{-iHT/\hbar} | x_a \rangle = \int \mathcal{D}x(t)\, e^{iS[x(t)]/\hbar}$$

Functional: map of functions to numbers.
Integral occurs over function space (i.e. trajectories $x(t)$), using measure $\mathcal{D}$.

Sketch QM procedure:
Transition from coordinate q_a to q_b.

Transition from coordinate q_a to q_b.

Label coordinates q^i + conjugate momenta: p^i

Hamiltonian: $H(q, p)$

$$U(q_a, q_b; T) = \langle q_b | e^{-iHT} | q_a \rangle$$

① Split T into N slices, $e^{iH\epsilon} \cdots e^{iH\epsilon}$

② Insert complete set of coordinate eigenstates.

$$\mathbb{1} = \int \prod_a dq_a \, |q\rangle\langle q| \qquad \mathbb{1} = \left(\prod_i \int dq^i_k \right) |q_k\rangle\langle q_k|$$

Repeat N-1 times:

$$= \left(\prod_i \int dp^i_k \right) |p_k\rangle\langle p_k|$$

get $\langle q_{k+1} | e^{-iH\epsilon} | q_k \rangle$

$$\simeq \langle q_{k+1} | 1 - iH\epsilon + \ldots | q_k \rangle$$

③ Operate H on eigenstates. Pure coordinates easy

Consider: $\langle q_{k+1} | f(q) | q_k \rangle = f(q_k) \prod_i \delta(q^i_k - q^i_{k+1})$

$$= f\left(\frac{q_{k+1} + q_k}{2} \right) \left(\prod_i \int \frac{dp^i_k}{(2\pi)} \right) \exp\left(i \sum p^i_k (q^i_{k+1} - q^i_k) \right)$$

$\langle q_{k+1} | f(p) | q_k \rangle$

$$= \left(\prod_i \int \frac{dp^i_k}{2\pi} \right) f(p_k) \exp\left(i \sum_i p^i_k (q^i_{k+1} - q^i_k) \right)$$

Using symmetrized form, $H(q, p)$ is then expressed as

$$\langle q_{k+1} | e^{-i\epsilon H} | q_k \rangle = \left(\prod_i \int \frac{dp^i_k}{(2\pi)} \right) \exp\left[-i\epsilon H\left(\frac{q_{k+1} + q_k}{2}, p_k \right) \right] \exp\left[i \sum_i p^i_k (q^i_{k+1} - q^i_k) \right]$$

Perform for N- inner products

$$U(q_0, q_N; T) = \left(\prod_{i,k} \int dq^i_k \int \frac{dp^i_k}{(2\pi)} \right) \exp\left[i \sum_k \left(\sum_i p^i_k (q^i_{k+1} - q^i_k) \right) - \epsilon H\left(\frac{q_{k+1} + q_k}{2}, p_k \right) \right]$$

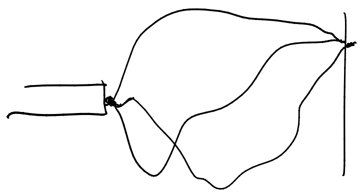
$N \to \infty$

Path integral in QM

$$= \left(\prod_i \int \mathcal{D}q(t)\, \mathcal{D}p(t) \right) \exp\left[i \int_0^T dt \left(\sum_i p^i \dot{q}^i - H(q,p) \right) \right]$$

$q(t)$ constrained, $p(t)$ is not constrained.

$q(t=0) = q_0$ + $q(t=T) = q_N$.



Now, consider quantum field theory.

Calculate correlation functions of fields.

Promote q^i to field amplitudes $\phi(x)$.

$$\langle \phi_b(\vec{x}) | e^{-iHT} | \phi_a(\vec{x}) \rangle$$
$$= \int \mathcal{D}\phi\, \mathcal{D}\pi \, \exp\left(i \int_0^T d^4x \left(\pi \dot{\phi} - \tfrac{1}{2}\pi^2 - \tfrac{1}{2}(\nabla\phi)^2 - V(\phi) \right) \right)$$

with $\phi(\vec{x}, 0) = \phi_a(\vec{x})$

+ $\phi(\vec{x}, T) = \phi_b(\vec{x})$

Complete the square + evaluate $\mathcal{D}\pi$ integral

$$\langle \phi_b(\vec{x}) | e^{-iHT} | \phi_a(\vec{x}) \rangle = \int \mathcal{D}\phi \, \exp\left[i \int_0^T d^4x\, \mathcal{L} \right]$$

$$\mathcal{L} = \tfrac{1}{2} (\partial_\mu \phi)^2 - V(\phi)$$

Correlation fcn. (ansatz)

$$\langle \Omega | T \phi(x_1) \phi(x_2) | \Omega \rangle$$
$$= \lim_{T \to \infty(1-i\epsilon)} \frac{\int \mathcal{D}\phi\, \phi(x_1)\phi(x_2) \exp\left[i \int_{-T}^{T} d^4x\, \mathcal{L} \right]}{\int \mathcal{D}\phi\, \exp\left[i \int_{-T}^{T} d^4x\, \mathcal{L} \right]}$$

Peskin + Schroeder 9.2

Derived from $\int \mathcal{D}\phi(x) = \int \mathcal{D}\phi_1(x) \int \mathcal{D}\phi_2(x) \int \mathcal{D}\phi(x)$

Derived from $\int D\phi(x) = \int D\phi_1(x) \int D\phi_2(x) \int D\phi(x)$

$\phi(x_1^0, \vec{x}) = \phi_1(\vec{x})$
$\phi(x_2^0, \vec{x}) = \phi_2(\vec{x})$

Higher correlation fcns.: add more factors.

Instead of adding more ϕ, do functional derivatives with source term that defines generating functional.

Define $\frac{\delta}{\delta J(x)} J(y) = \delta^{(4)}(x-y)$

$$\frac{\delta}{\delta J(x)} \int d^4y \, J(y)\phi(y) = \phi(x)$$

$$\frac{\delta}{\delta J(x)} \exp\left[i\int d^4y \, J(y)\phi(y)\right] = i\phi(x)\exp\left[i\int d^4y \, J(y)\phi(y)\right]$$

$$\frac{\delta}{\delta J(x)} \int d^4y \, \partial_\mu J(y) V^\mu(y) = \underbrace{-\partial_\mu V^\mu(x)}_{\text{IBP}}$$

Define: generating functional

$$\star \quad Z[J] \equiv \int D\phi \, \exp\left[i\int d^4x \left[\mathcal{L} + J(x)\phi(x)\right]\right]$$

Then n-pt. correlation fcns.

$$\langle \Omega | T\phi(x_1)\ldots\phi(x_n) | \Omega \rangle = \frac{1}{Z_0}\left(\frac{-i\delta}{\delta J(x_1)}\right)\left(\frac{-i\delta}{\delta J(x_2)}\right)\cdots\left(\frac{-i\delta}{\delta J(x_n)}\right) Z[J]\Big|_{J=0}$$

$$Z_0 = Z[J=0]$$

Discussion:

How to interpret $Z[J]$?

J acts as an arbitrary source term.

Evaluating response of system (for some n) + then remove the source.

Think in terms of Lagrange multipliers.

J has no dynamics. Can replace by EOM for J.

n-pt. corr. fcn. is like Taylor series for EOM of J.

[Q: how to see quantization of quantum field ϕ?]

Cheng & Li, Appendix δ.

Free $\mathcal{L}_0 = \frac{1}{2}(\partial_\mu \phi)^2 - \frac{1}{2}\mu^2\phi^2$

$\mathcal{L}_{int} = -\frac{\lambda}{4!}\phi^4$

$$W[J] = \int [d\phi] \exp\left[i\int (\mathcal{L}_0 + \mathcal{L}_{int} + J\phi) d^4x\right]$$

$$= \exp\left[i\int d^4x\, \mathcal{L}_I\left(\frac{-i\partial}{\partial J}\right)\right] W_0[J]$$

argument replaced by functional deriv.

$$W_0[J] = \int [d\phi] \exp\left[i\int d^4x\, (\mathcal{L}_0 + J(x)\phi(x))\right]$$

Solve free theory:

Green's fcns. from K-G equation.

$$\Delta_F(k) = \frac{1}{k^2 - \mu^2 + i\epsilon}$$

Continue next time.