Lecture 7, May 13, 2020

Monday, May 11, 2020 11:04 AM

Last time: Continued discussion of Callan-Symanzik equation in $\lambda\phi^4$ theory.
Introduced $\beta$-fcn and anomalous dimension.

Today: More complete refresher on QED renormalization
Asymptotic behavior in UV + IR,
running of couplings.

☆ May not get to functional methods, so will postpone Homework 2-3 until Homework 3.

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Review of QED renormalization:

P+S Section 10.3.

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \bar{\psi}(i\not{\partial} - m_0)\psi - e\bar{\psi}\gamma^\mu \psi A_\mu$$

Procedure: ① Insert WFR factors for fields.

[This was motivated $|\Omega\rangle \neq |0\rangle$, $\langle\Omega|\phi|p\rangle = \sqrt{Z}$]

$$\psi = \sqrt{Z_2}\,\psi_R \quad , \quad A^\mu = \sqrt{Z_3}\, A^\mu_R$$

②
$$\mathcal{L} = -\frac{1}{4} Z_3 F_{\mu\nu_R} F^{\mu\nu}_R + \bar{\psi}_R\, i\not{\partial} Z_2 \psi_R - m_0 Z_2 \bar{\psi}_R \psi_R - e_0 Z_2 \sqrt{Z_3}\, \bar{\psi}_R \gamma^\mu \psi_R A_{\mu_R}$$

Expand WFR factors into separate counterterms.
Also, separate renormalized coupling constants + masses from the bare parameters, which are rescaled by WFR factors.

Ex. $\mathcal{L} \supset -m_0 \bar{\psi}\psi = -m_0 Z_2 \bar{\psi}_R \psi_R$

$$= -(m + \delta_m)\bar{\psi}_R \psi_R$$

$$= -m\bar{\psi}_R \psi_R - \delta_m \bar{\psi}_R \psi_R$$

$$\mathcal{L} \supset -e_0 Z_2 \sqrt{Z_3}\, \bar{\psi}_R \gamma^\mu \psi_R A_{\mu R}$$

$$\equiv -e Z_1 \bar{\psi}_R \gamma^\mu \psi_R A_{\mu R}$$

We get a Lagrangian with a single $Z$ for each term:

$$\mathcal{L} \supset -\frac{1}{4} Z_3 F_{\mu\nu R} F^{\mu\nu}_R + Z_2 \bar{\psi}_R i\not{\partial}\psi_R - m\bar{\psi}_R \psi_R - e Z_1 \bar{\psi}_R \gamma^\mu A_{\mu R}\psi_R - \delta_m \bar{\psi}_R \psi_R$$

We now split off the counterterm from each $Z_i$:

$$Z_i \equiv 1 + \delta_i$$

$$\mathcal{L} \supset -\frac{1}{4} F_{\mu\nu_R} F^{\mu\nu}_R + \bar{\psi}_R i\not{\partial}\psi_R - m\bar{\psi}_R\psi_R - e\bar{\psi}_R \gamma^\mu A_{\mu R}\psi_R$$
$$-\frac{1}{4}\delta_3 F_{\mu\nu_R} F^{\mu\nu}_R + \delta_2 \bar{\psi}_R i\not{\partial}\psi_R - \delta_m \bar{\psi}_R\psi_R - e\delta_1 \bar{\psi}_R\gamma^\mu A_{\mu R}\psi_R$$

(Could have done $Z_m = 1 + \delta_m$,
$-(m Z_m) = -(m + \delta_m)$ )

Emphasize:

The first line is the same as the original bare Lagrangian, except that all fields, masses, + couplings are renormalized values. If you ignored the second line, would have same local description as original Lagrangian.

The second line are all counterterms, which are all formally infinite. The easiest way to conceptualize the counterterms is that they provide an infinite offset to the local Lagrangian given by the first line: this does not change the local description or dynamics, but they render the perturbative calculation finite order by order.

The counterterms are fixed such that the overall Lagrangian gives finite answers for a set of physical observables: these are called renormalization conditions.

Adopt notation

Expand to $O(e^n)$


$= i\Pi^{\mu\nu}(q) = i(g^{\mu\nu}q^2 - q^\mu q^\nu)\Pi(q^2)$


$= -i\Sigma(\not{p})$


$= -ie\Gamma^\mu(p', p)$

Ren. conds. of QED

$$\Sigma(\not{p} = m) = 0$$
$$\frac{d}{d\not{p}}\Sigma(\not{p})\Big|_{\not{p}=m} = 0$$
$$\Pi(q^2 = 0) = 0$$
$$-ie\Gamma^\mu(p' - p = 0) = -ie\gamma^\mu$$

We can calculate each of these 1PI diagrams in pertubation theory at a given order. Note that the renorm. conds. are unchanged no matter how high the renorm. conds. are unchanged no matter how high in pert. theory you calculate: by the BPNZ theorem, we do not need more counterterms for this <u>renormalizable</u> theory.

Go through the calculation in Sec. 10.3 to evaluate all 1PI diagrams to 1-loop order. Imposing ren. conditions fixes counterterms $\delta_1, \delta_2, \delta_3, \delta_m$. In dim. reg., the divergences are seen as $\frac{1}{\epsilon}$ singularities.

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$\beta$-fcn. and anomalous dimension $\gamma$.

Layman's terms: Renormalization conditions can be thought of as physical measurements, prescribing definition of $\infty - \infty$ at some scale $\mu_1$ where you make measurement. A second person can choose a second scale $\mu_2$, where they make measurements; the definition of $\infty - \infty$ is not the same since $\mu_1 \neq \mu_2$. But if I make a prediction for my result at $\mu_2$ + measure at $\mu_2$, must converge exactly to the physical results of second person. The physics of a given system flows in a non-trivial way between different energy scales: this flow is called the renormalization group flow, and the ODE for the flow is the Callen-Symanzik equation.

$(\infty - \infty)_{\mu_1} = \text{finite}_1$

$(\infty - \infty)_{\mu_2} = \text{finite}_2$

once I have finite$_1$, I can predict what to measure at $\mu_2$, + I should <u>predict</u> finite$_2$.

The important terms in C-S eqn:

$\mu \frac{\partial}{\partial \mu} G^{(n)}$ : change in overall correlation fcn. as a fcn. of $\mu$.

$\beta$-fcn: $\mu \frac{\partial \lambda}{\partial \mu}$ or $\mu \frac{\partial g}{\partial \mu}$ for some coupling $\lambda$ or $g$.

Change in the coupling constant as fcn. of $\mu$.

Change in the coupling constant as fcn. of $\mu$.

$\gamma$: anomalous dimension $= \frac{1}{n}\frac{\mu}{Z}\frac{\partial Z}{\partial \mu}$

Deviation from classical scaling behavior of $G_n$.

Schwartz: 23.4.4.

$[\phi] = 1, \quad [\psi] = 3/2, \quad [A_\mu] = 1$

$$S = \int d^4x \left[ -\frac{1}{2}(\partial\phi)^2 + \frac{m^2\phi^2}{2} + g\phi^4 \right]$$

Dilatation operator: $x^\mu \to \frac{1}{\lambda}x^\mu$, $\partial_\mu \to \lambda\partial_\mu$, $m \to \lambda m$

$g \to g$, $\phi \to \lambda\phi$.

$$G_n = \langle \Omega | T\{\phi_1(x_1) \ldots \phi_n(x_n)\} | \Omega \rangle$$

Classically, $G_n(x, g, m) = m^a g^b x_1^{c_1} \ldots x_n^{c_n}$

subject to $a - c_1 - c_n = n$.

Quantum mechanically, $G_n(x, g, m, \mu) = m^a g^b x_1^{c_1} \ldots x_n^{c_n} \mu^\gamma$

$a - c_1 - c_n = n - \gamma$.

Rescaled by dilatation, $G_n \to \lambda^n G_n$, classically,

but QM, $G_n \to \lambda^{n-\gamma} G_n$. So $\gamma$ is the difference

= anomalous dimension of the scaling behavior of $G^{(n)}$ from classical behavior.

Simple example:

$$G^{(4)} = m^2\phi^2$$

$m \to 2m$

$$G^{(4)\prime} = 4m^2\phi^2 \quad \text{or} \quad 4^{1-0.1} m^2\phi^2$$

Perhaps, as I make $m$ larger ($m = 1\,\text{GeV} \to 100\,\text{GeV} \to 10^{16}\,\text{GeV}$)

might run into phase boundary, and lose scaling behavior with $m$.

See Lecture 6 notes for connection b/t $\beta$ & $\gamma$ with the coefficients of UV divergences.

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Asymptotic behavior.

St... [illegible]

Asymptotic behavior.

Study $\beta(\lambda) = \frac{3\lambda^2}{16\pi^2} = \mu \frac{\partial \lambda}{\partial \mu}$ from $\lambda \phi^4$ theory.

Solve for evolution eqn. of $\lambda$ wrt. $\mu$

$$\frac{3}{16\pi^2} \frac{\partial \mu}{\mu} = \frac{\partial \lambda}{\lambda^2}$$

$$\frac{3}{16\pi^2} \ln \mu + C = -\frac{1}{\lambda}$$

Absorb integration const. $C$ as scale $\mu_0$ where we measure $\lambda_0$

$$\frac{3}{16\pi^2} \ln \frac{\mu}{\mu_0} = \frac{-1}{\lambda} + \frac{1}{\lambda_0}$$

$$\frac{1}{\lambda} = \frac{1}{\lambda_0} - \frac{3}{16\pi^2} \ln \frac{\mu}{\mu_0} = \frac{1}{\lambda_0}\left(1 - \frac{3\lambda_0}{16\pi^2} \ln \frac{\mu}{\mu_0}\right)$$

$$\lambda = \frac{\lambda_0}{1 - \underbrace{\frac{3\lambda_0}{16\pi^2} \ln \frac{\mu}{\mu_0}}}$$

less than 1;

Predicted coupling $\lambda$ as fcn. $\mu$, given $\lambda(\mu = \mu_0) = \lambda_0$.

As $\mu$ grows, $\lambda$ grows (positive $\beta$-fcn.) in UV.

In fact, will have scale where pertubativity breaks down. Scale is called Landau pole.

[Only consistent solution to this behavior is $\lambda_0 = 0$]

"triviality of $\lambda\phi^4$"

It is also possible to have the opposite behavior when $\beta$-fcn. is negative.

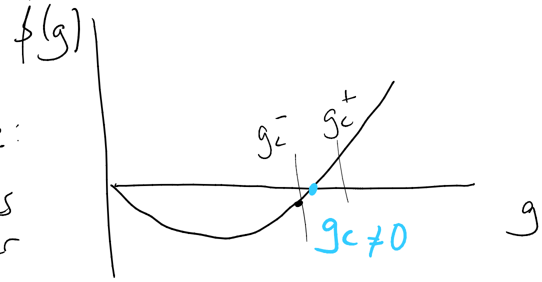
Example: Quantum Chromodynamics (in SM) has neg. $\beta$-fcn. As we go to higher energies, gauge coupling $g_s$ of QCD becomes weaker and approaches 0. Called "asymptotic freedom."

Can also have $\beta = 0$ (or $\beta \simeq 0$).

Conformal theories, since gauge coupling (or interaction

Conformal theories, since gauge coupling (or interaction coupling) does not change with energy (or at least changes very little.) Occur at fixed points in theories:

$\beta(g)$ positive: $g$ increases with larger energy.

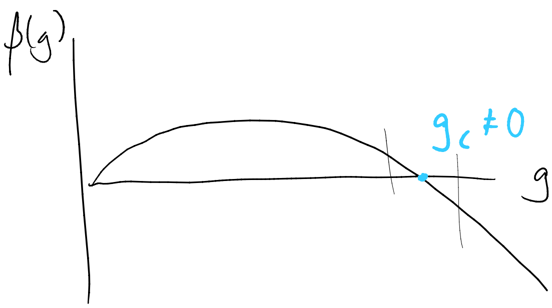



At $g_c^-$, smaller energy drives me toward $g_c$.

At $g_c^+$, smaller energy drives me toward $g_c$.

Combine to IR fixed point.

$\beta(g)$ negative: $g$ decreases with larger energy.




UV fixed point.

interacting!

Near $g_c$, $\beta \simeq -B(g-g_c)$, $B>0$

$$\frac{dg}{d\log \mu/\mu_0} \simeq -\beta(g-g_c)$$

Soln: $g(\mu) \simeq g_c + C\left(\frac{\mu_0}{\mu}\right)^{\beta}$