Lecture 7.

QFT II.

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Last time: Callan-Symanzik equation in $\lambda\phi^4$ theory.

Begin with review of QED renormalization

Follow P+S section 10.3.

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu}F^{\mu\nu} + \bar{\psi}(i\not{\partial} - m_0)\psi - e_0 \bar{\psi}\gamma^\mu \psi A_\mu$$

Procedure: 1 Insert wavefcn. renormalizations

$$\psi = \sqrt{Z_2}\,\psi_R, \quad A^\mu = \sqrt{Z_3}\,A^\mu_R$$

Expand the wavefcn. factors to separate counterterms. Also separate renormalized coupling constants and masses from bare parameters which are rescaled by WFR factors.

For example, $\mathcal{L} \supset -m_0 \bar{\psi}\psi = -m_0 Z_2 \bar{\psi}_R \psi_R$

$$= -(m + \delta_m)\bar{\psi}_R \psi_R$$

$$= -m\bar{\psi}_R\psi_R - \delta_m \bar{\psi}_R \psi_R$$

Another example, $\mathcal{L} \supset -e_0 \bar{\psi}_R \gamma^\mu \psi_R A_{\mu R} \cdot Z_2\sqrt{Z_3}$

$$\equiv -e Z_1 \bar{\psi}_R \gamma^\mu \psi_R A_{\mu R}$$

We get a Lagrangian with <u>one</u> Z for each term:

$$\mathcal{L} = -\frac{1}{4} Z_3 F_{\mu\nu R}F^{\mu\nu}_R + \bar{\psi}_R Z_2\, i\not{\partial}\psi - m\bar{\psi}_R\psi_R - e Z_1 \bar{\psi}_R \gamma^\mu A_{\mu R}\psi_R - \delta_m \bar{\psi}_R\psi_R$$

We now split off the counterterm from each Z_i:

$$Z_i = 1 + \delta_i.$$

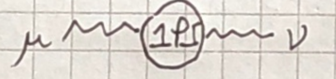
$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu R}F^{\mu\nu}_R + \bar{\psi}_R i\not{\partial}\psi_R - m\bar{\psi}_R\psi_R - e\bar{\psi}_R\gamma^\mu A_{\mu R}\psi_R$$
$$-\frac{1}{4}\delta_3 F_{\mu\nu R}F^{\mu\nu}_R + \delta_2\bar{\psi}_R i\not{\partial}\psi_R - \delta_m\bar{\psi}_R\psi_R - e\delta_1\bar{\psi}_R\gamma^\mu A_{\mu R}\psi_R$$

Emphasize:

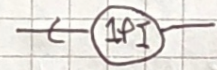
The first line is the same as the original bare Lagrangian, except all fields + masses + couplings are now renormalized values.

The second line are all counterterms, which are all formally infinite. The easiest way to conceptualize the counterterms is that they add an infinite offset to the local Lagrangian: this does not change the local description or dynamics, which are encoded by the first line, but instead they render the perturbative calculation finite order by order.

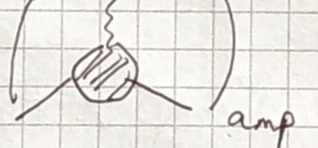
The counterterms are fixed such that the overall Lagrangian gives finite answers for a set of physical observables: these are the renormalization conditions:



$$= i\Pi^{\mu\nu}(q) = i(g^{\mu\nu}q^2 - q^\mu q^\nu)\,\Pi(q^2)$$



$$= -i\Sigma(\not{p})$$



$$= -ie\Gamma^\mu(p', p)$$

Gives these conditions
$$\Sigma(\not{p} = m) = 0$$
$$\frac{d}{d\not{p}}\Sigma(\not{p})\Big|_{\not{p}=m} = 0$$
$$\Pi(q^2 = 0) = 0$$
$$-ie\Gamma^\mu(p'-p=0) = -ie\gamma^\mu$$

We can calculate each of these 1PI diagrams in perturbation theory. Note that the renormalization conds. are unchanged no matter how high in pert. theory we calculate: by the BPHZ theorem, we do not need more counterterms for this renormalizable theory.

With the introduction of β-fcns, we now turn to the question of running couplings and asymptotic behavior.

Recall that the β-fcn. for $\lambda\phi^4$ theory was

$$\beta(\lambda) = \frac{3\lambda^2}{16\pi^2}, \text{ with } \mu\frac{\partial\lambda}{\partial\mu} = \beta(\lambda)$$

We can solve the diff. eqn. for λ at another scale, with the boundary condition $\lambda(\mu=\mu_0) = \lambda_0$.

$$\frac{3\lambda^2}{16\pi^2} = \mu\frac{\partial\lambda}{\partial\mu} \Rightarrow \frac{3\,\partial\mu}{16\pi^2\mu} = \frac{\partial\lambda}{\lambda^2}$$

$$\frac{3\ln\mu}{16\pi^2} = -\frac{1}{\lambda}$$

or

$$\frac{3}{16\pi^2}\ln\frac{\mu}{\mu_0} = -\frac{1}{\lambda} + \frac{1}{\lambda_0} = \frac{-\lambda_0 + \lambda}{\lambda\lambda_0}$$

$$\frac{1}{\lambda} = \frac{1}{\lambda_0} - \frac{3}{16\pi^2}\ln\frac{\mu}{\mu_0} = \frac{1}{\lambda_0}\left(1 - \frac{3\lambda_0}{16\pi^2}\ln\frac{\mu}{\mu_0}\right)$$

$$\boxed{\lambda = \frac{\lambda_0}{\left(1 - \frac{3\lambda_0}{16\pi^2}\ln \mu/\mu_0\right)}}$$

As μ grows, coupling grows (positive β).
(Implicitly have $\frac{3\lambda_0}{16\pi^2} \ll 1$ for perturbativity.)

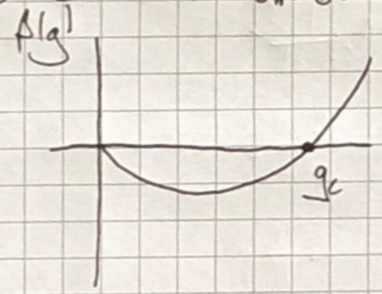
At some scale, we have $\frac{3\lambda_0}{16\pi^2}\ln\frac{\mu}{\mu_0} \simeq 1$, when the corresponding λ becomes non-perturbative. The scale where this happens is called a Landau pole.
(As an aside, the only consistent solution for perturbativity to be valid everywhere is to set $\lambda_0 = 0$, which implies that $\lambda\phi^4$ theory is <u>trivial</u> [no $\lambda\phi^4$ coupling at all].)

We can also imagine that $\beta < 0$ for another theory, for example quantum chromodynamics. Then the coupling get weaker in the UV: this is called asymptotic freedom.

(~~G~~ Gross, Politzer, Wilczek won the 2004 Nobel Prize for this discovery.)

As a last category, we can have $\beta \approx 0$ for some energy range. As a result, the coupling constant changes very slowly or not at all. This is called a(n approximately) conformal theory.
For $\beta(g_c) \approx 0$ for some critical value of the coupling g_c, we can have two types of trajectories:

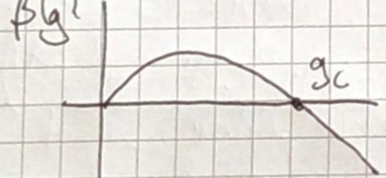
Case 1: IR-stable fixed point



Recall $\beta(g) > 0$ implies that g grows as scale increases, while $\beta(g) < 0$ implies that g decreases as scale decreases.

In this case, for $g > g_c$, if we decrease the scale, then $g \to g_c$. For $g < g_c$, if we decrease the scale, then $g \to g_c$. Thus, as we go to lower ~~scale~~ energies, we have $g \to g_c$ in either direction. This is an <u>IR</u> (infrared) stable fixed point.

Case 2: UV fixed point



Here, for $g < g_c$ if we increase the energy scale, then $g \to g_c$. For $g > g_c$, if we increase the scale, then $g \to g_c$ also. So as we raise the scale, we always go towards g_c.

Near g_c, $\beta \simeq -b(g-g_c)$, $b>0$

$$\frac{dg}{d\log \mu/\mu_0} \simeq -b(g-g_c)$$

Solution: $g(\mu) \simeq g_c + C\left(\frac{\mu_0}{\mu}\right)^b$ As $\mu \to \infty$, $g \to g_c$

Consider $G(\mu)$ solution to Callan-Symanzik eqn.

$$G^{(2)}(\mu,\lambda) = \hat{G}(\bar{\lambda}(\mu,\lambda))\exp\left(-\int_{p'=\mu_0}^{p'=\mu} d\log\left(\frac{p'}{\mu_0}\right) 2(1-\gamma(\bar{\lambda}))\right)$$

with ~~dlog~~ $\frac{d}{d\log\left(\frac{\mu}{\mu_0}\right)}\bar{\lambda} = \beta(\bar{\lambda})$

and $\bar{\lambda}(\mu_0) = \lambda$

$\hat{G}$ is unknown, must match order by order in PT.

At $\mu \to \infty$ & fixed UV point,

$G^{(2)}(\mu,\lambda)$ dominated by $\bar{\lambda}$ close to λ_c.

$$G^{(2)}(\mu,\lambda) \simeq \hat{G}(\lambda_c)\exp\left(-\log\left(\frac{\mu}{\mu_0}\right)2(1-\gamma(\lambda_c))\right)$$
$$\simeq C\left(\frac{1}{\mu^2}\right)^{1-\gamma(\lambda_c)}$$

So 2-pt. correlation fcn. becomes simple scaling law, but power law deviates from NDA expectation by $\gamma(\lambda_c)$. γ is anomalous dimension, reflects how interactions of theory affect the law of rescaling.

Other interesting topics:

Universality of large logs: Schwartz, 23.1

Coleman-Weinberg (HW3)

Dimensional transmutation (HW3)

Can calculate behavior of theories, check measurements across scales. [Need to account for mass thresholds.]