Lecture 6, May 11, 2020

Friday, May 8, 2020 10:28 AM

Last time:

Renormalization group and $\lambda\phi^4$ theory.

Callan-Symanzik equation

Today: Continue with C-S equation

Understand β-fcn. and anomalous dimension coefficients

Asymptotic behavior in UV + IR.

We had $\mathcal{L} \supset \frac{\lambda}{4!}\phi^4$ theory. Renorm. cond. of ()$_{\text{amp.}} = -i\lambda$

At one-loop, counterterm defined to cancel all divergences in s-channel, t-channel, u-channel diagrams. Ended up with manifestly finite matrix element, imposed ren. cond. at $s = 4m^2$, $t=0$, $u=0$.

$$i\mathcal{M} = -i\lambda + \frac{i\lambda^2}{32\pi^2}\int_0^1 dx \log\left(\frac{x^2 s - xs + m^2}{4x^2m^2 - 4xm^2 + m^2}\right)$$
$$+ \frac{i\lambda^2}{32\pi^2}\int_0^1 dx\left[\log\left(\frac{x^2 t - xt + m^2}{m^2}\right) + \log\left(\frac{x^2 u - xu + m^2}{m^2}\right)\right]$$

Separately, considered what is meaning of scale where we set ren. condition?

For simplicity, set scale $s = t = u = -\mu^2$. [Avoid poles] for ren. conditions.

This gives $i\delta_\lambda = \frac{i\lambda^2}{32\pi^2}\int_0^1 dx \left(\frac{6}{\epsilon} - \log\Delta_s - \log\Delta_t - \log\Delta_u - 3\gamma + 3\log 4\pi\right)$

$$\Delta_v = x^2 v - xv + m^2$$

We get (take $m \to 0$)

$$i\delta_\lambda = \frac{i\lambda^2}{32\pi^2}\int_0^1 dx\left(\frac{6}{\epsilon} - 3\log\left(-x^2\mu^2 + x\mu^2\right) - 3\gamma + 3\log 4\pi\right)$$

$$= \frac{i3\lambda^2}{32\pi^2}\int_0^1 dx\left(\frac{2}{\epsilon} - \log\mu^2 + \text{finite [no } \mu \text{ dependence]}\right)$$

Since Feynman rule had $-i\delta_\lambda$,

$$\left[\mu\frac{\partial G^{(4)}}{\partial\mu} = -\mu\left(\frac{3\lambda^2}{32\pi^2}\cdot\frac{-1}{\mu^2}\cdot 2\mu\right) = \frac{+3\lambda^2}{16\pi^2}\right]$$

Recall Callan-Symanzik equation.

$$0 = \Bigg[n\underbrace{\left(\frac{1}{2}\frac{\mu}{Z}\frac{\partial Z}{\partial\mu}\right)}_{\boxed{\gamma(\lambda)}\ \text{anom. dim.}}\underbrace{G_n}_{\theta(\lambda)} + \underbrace{\mu\frac{\partial\lambda}{\partial\mu}}_{\beta(\lambda)\ \beta\text{-fcn.}}\frac{\partial G_n}{\partial\lambda} + \mu\frac{\partial G_n}{\partial\mu}\Bigg]$$

Differential equation bare + renormalized Green's fcn. wrt. μ.

Reflects physical behavior of renormalized Green's fcn. if you chose different renormalization scale μ.

Reflects physical behavior of renormalized Green's fcn. if you chose different renormalization scale μ.

Apply perturbative expansion, we can match power series expansions in each term of C-S eqn., get matched diff. eq. order-by-order.

Will claim $\gamma(\lambda)$ has no term at $O(\lambda)$,
(note G_n starts at $O(\lambda)$, so we have $\gamma(\lambda)G_n$ starting at $O(\lambda^2)$).

So, $\mu \frac{\partial G^{(4)}}{\partial \mu} = \frac{3\lambda^2}{16\pi^2}$ gives $\beta(\lambda) = \frac{3\lambda^2}{16\pi^2}$, next term is $O(\lambda^3)$.

$$\frac{\partial G^{(4)}}{\partial \lambda} = -1$$

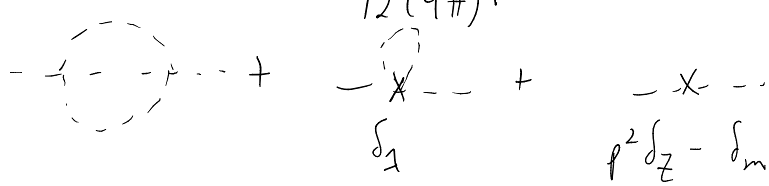
$$\mu \frac{\partial G^{(4)}}{\partial \mu} = -\beta(\lambda)\frac{\partial G^{(4)}}{\partial \lambda}$$

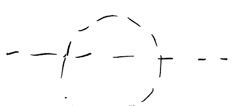
$$\frac{3\lambda^2}{16\pi^2} = -\beta(\lambda)\cdot(-1)$$

$$\frac{3\lambda^2}{16\pi^2} = \beta(\lambda)$$

Can calculate $\gamma(\lambda) = \frac{\lambda^2}{12(4\pi)^4}$

from



 $= -ip^2 \frac{\lambda^2}{12(4\pi)^4}\left(\frac{-1}{\epsilon} + \log p^2 + \ldots\right)$

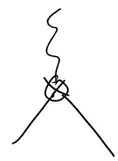
Aside: In general, $\gamma(g)G + \beta(g)\frac{\partial G}{\partial g}$ start at same order.

Another example: QED

P+S, section 10.3. (p. 332).

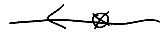
Counterterms were:

δ_1 = vertex counterterm, gave Feynman rule



$-ie\gamma^\mu \delta_1$

δ_2 = fermion propagator counterterm + δ_m = mass

 $i(\not{p}_2\delta_2 - \delta_m)$

δ_3 = WFR for photon

 $-i(g^{\mu\nu}q^2 - q^\mu q^\nu)\delta_3$

Impose ren. conditions at scale $-\mu^2$,

$$\delta_1 = \delta_2 = \frac{-e^2}{4\pi^2}\frac{\Gamma(2-\frac{d}{2})}{(\mu^2)^{2-d/2}} + \text{finite}$$

$$\delta_3 = \frac{-e^2}{4\pi^2}\cdot\frac{4}{3}\frac{\Gamma(2-\frac{d}{2})}{(\mu^2)^{2-d/2}} + \text{fin.}$$

Applying C-S equation,

$$\beta(e) = \mu\frac{d}{d\mu}\Big(-e\delta_1 + \underbrace{e\delta_2}_{\text{anom. dim. of e}} + \underbrace{\tfrac{e}{2}\delta_3}_{\text{anom. dim. of photon}}\Big)$$

$$= \frac{e^3}{12\pi^2}$$

Back up:

4 renormalization conditions

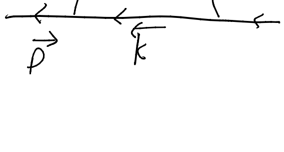
$\Sigma(\not{p} = m) = 0$ (no shift in pole)

$\frac{d}{d\not{p}}\Sigma(\not{p})\Big|_{\not{p}=m} = 0$ (residue is unchanged)

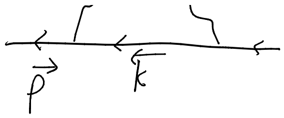
$\Pi(q^2=0) = 0$ (residue is unchanged)

$-ie\Gamma^\mu(p'-p=0) = -ie\gamma^\mu$

Calculate 1PI correction to electron self-energy.



$$-i\Sigma_2(p) = \frac{-ie^2}{(4\pi)^{d/2}}\int_0^1 dx\,\frac{\Gamma(2-\frac{d}{2})}{([illegible] - x(1-x)p^2)^{2-d/2}}$$



$$-i\Sigma_2(p) = -i\frac{e^2}{(4\pi)^{d/2}}\int_0^1 dx \, \frac{\Gamma(2-\frac{d}{2})}{\left((1-x)m^2 + x\mu^2 - x(1-x)p^2\right)^{2-d/2}}$$

P+S, eq. 10.41.

$$\cdot\left((4-\epsilon)m - (2-\epsilon)x\not p\right)$$

Apply first two ren. conditions

① $$m\delta_2 - \delta_m = \Sigma_2(m) = \frac{e^2 m}{(4\pi)^{d/2}}\int_0^1 dx \, \frac{\Gamma(2-\frac{d}{2})\left(4-2x-\epsilon(1-x)\right)}{\left((1-x)^2 m^2 + x\mu^2\right)^{2-d/2}}$$

② $$\delta_2 = \frac{-e^2}{(4\pi)^{d/2}}\int_0^1 dx \, \frac{\Gamma(2-\frac{d}{2})}{\left((1-x)^2 m^2 + x\mu^2\right)^{2-d/2}}$$

$$\cdot\left[(2-\epsilon)x - \frac{\epsilon}{2}\frac{2x(1-x)m^2}{(1-x)^2m^2 + x\mu^2}\left(4-2x-\epsilon(1-x)\right)\right]$$

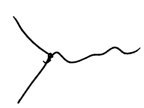
Third ren. cond.

Photon self-energy

$$\Pi_2(q^2) = \frac{-e^2}{(4\pi)^{d/2}}\int_0^1 dx \, \frac{\Gamma(2-\frac{d}{2})}{\left(m^2 - x(1-x)q^2\right)^{2-d/2}} \cdot \left(8x(1-x)\right)$$

③ Evaluate at $q^2=0$

$$\delta_3 = \Pi_2(0) = \frac{-e^2}{(4\pi)^{d/2}}\int_0^1 dx \, \frac{\Gamma(2-\frac{d}{2})}{(m^2)^{2-d/2}}\left(8x(1-x)\right)$$

Fourth: calculate form factor vertex fcn.  (q)

$F_1\gamma^\mu$ and $F_2\sigma^{\mu\nu}\gamma_\nu$.

Break ~ 10 min.

Resume w/ electron vertex in QED.

(Important) Long aside: Section 6.3 of P+S.

This is the 1-loop vertex correction in QED.

$$iM = \bar{u}(p')\int d^4k \; (-ie\gamma^\nu)\, i(\not k + \not q + m)\; (-ie\gamma^\mu)\, i(\not k + m)\; (-ie\gamma^\rho)\, u(p)$$

L ρ

$$iM = \bar u(p')\int \frac{d^4k}{(2\pi)^4}\,(-ie\gamma^\nu)\,\frac{i(\not k+\not q+m)}{(k+q)^2-m^2}\,(-ie\gamma^\mu)\,\frac{i(\not k+m)}{k^2-m^2}\,(-ie\gamma^\rho)\,u(p)\;\frac{-ig_{\nu\rho}}{(p-k)^2}$$

$$= \int\frac{d^4k}{(2\pi)^4}\,-e^3\,\frac{\bar u(p')\gamma^\nu(\not k+\not q+m)\gamma^\mu(\not k+m)\gamma_\nu u(p)}{((k+q)^2-m^2)(k^2-m^2)(k-p)^2}$$

Use Dirac eqn. on spinors

$$= \int\frac{d^4k}{(2\pi)^4}\,-2e^3\,\frac{\bar u(p')\left[\not k\gamma^\mu(\not k+\not q)+m^2\gamma^\mu-2m(k+k')^\mu\right]u(p)}{D_1D_2D_3}$$

$\gamma^\nu\gamma^\mu\gamma_\nu = -2\gamma^\mu$

Numerator: Not just γ^μ structures.

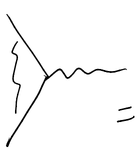
$$\bar u(p')\left[\not k\gamma^\mu(\not k+\not q)+m^2\gamma^\mu-2m(k+k')^\mu\right]u(p)$$

$$\to \bar u(p')\Big[-\tfrac{1}{2}\gamma^\mu\ell^2+(-y\not q+z\not p)\gamma^\mu((1-y)\not q+z\not p)$$
$$+m^2\gamma^\mu-2m((1-2y)q^\mu+2zp^\mu))\Big]u(p)$$

Possible structures: Lorentz invariance:

$$\underbrace{\gamma^\mu A+(p'^\mu+p^\mu)B}_{\substack{F_1\\ \text{Electric}}}+\underbrace{q^\mu C}_{\substack{F_2\,\propto\,\sigma^{\mu\nu}q_\nu\\ \text{Magnetic}}}$$

P+S Eq. 6.54



$$= \frac{\alpha}{2\pi}\int_0^1 dx\,dy\,dz\,\delta(x+y+z-1)$$
$$\bar u(p')\left(\gamma^\mu\left[\log\frac{z\Lambda^2}{\Delta}+\frac{1}{\Delta}\left((1-x)(1-y)q^2+(1-4z+z^2)m^2\right)\right]+\frac{\sigma^{\mu\nu}q_\nu}{2m}\left[\frac{1}{\Delta}2m^2z(1-z)\right]\right)u(p)$$

4th renormalization condition in QED

$$-ie\Gamma^\mu(p'-p=0) = -ie\gamma^\mu$$

$$\delta F_1(q^2) = \frac{e^2}{(4\pi)^{d/2}}\int dx\,dy\,dz\,\delta(x+y+z-1)\left[\frac{\Gamma(2-d/2)}{\Delta^{(2-d/2)}}\cdot\frac{(2-\epsilon)^2}{2}\right.$$

$$\delta F_1(q^2) = \frac{e^2}{(4\pi)^{d/2}} \int dx\,dy\,dz\; \delta(x+y+z-1) \left[\frac{\Gamma(2-\frac{d}{2})}{\Delta^{(2-d/2)}} \cdot \frac{(2-\epsilon)^2}{2} + \frac{\Gamma(3-\frac{d}{2})}{\Delta^{3-d/2}} \left(q^2\left(2(1-x)(1-y) - \epsilon xy\right) + m^2\left(2(1-4z+z^2) - \epsilon(1-z)^2\right)\right)\right]$$

$$\Delta \equiv (1-z)^2 m^2 + z\mu^2 - xyq^2$$

$$\delta_1 = -\delta F_1(q^2=0).$$

Generalization of β-fcn $+\,\gamma$. Where do they come from in general?

P+S, 12.2, p. 413

Massless, 2pt. Green's fcn.

$$G^{(2)}(p) = \text{---} \cdots + \text{loop} + \text{---}\otimes\text{---} + \cdots$$

div. finite shift

$$= \frac{i}{p^2} + \frac{i}{p^2}\left(A \log\frac{\Lambda^2}{-p^2} + \text{finite}\right)\frac{i}{p^2} + \frac{i}{p^2}\left(p^2\delta_Z\right)\frac{i}{p^2}$$

const.

As a result,

$n=2$ for $G^{(2)}$

$$\frac{-i}{p^2}\,\mu\frac{\partial}{\partial\mu}\delta_Z + 2\gamma\frac{i}{p^2} = 0$$

Gives $\gamma = \frac{1}{2}\mu\frac{\partial}{\partial\mu}\delta_Z$

Explicitly, $\delta_Z = A\log\frac{\Lambda^2}{\mu^2} + \text{finite}$

So $\gamma = -A$ to lowest order.

Anomalous dimension matches the factor A in the UV divergence of the loop integral.

Also, $G^{(n)} = \text{tree} + \text{1PI loop} + \text{vtx ct.} + \text{external leg}$

$$= -ig - iB\log\frac{\Lambda^2}{-p^2} - i\delta_g + (-ig)\left[\sum_i\left(A_i\log\frac{\Lambda^2}{-p_i^2} - \delta_{Z_i}\right) + \text{finite}\right]$$

Get $$\mu\frac{\partial}{\partial\mu}\left(\delta_g - g\sum\delta_{Z_i}\right) + \beta(g) + g\sum_i\frac{1}{2}\mu\frac{\partial}{\partial\mu}\delta_{Z_i} = 0$$

From C-S, $\beta(g) = \mu\,\partial\;(\;\ldots$

$$\cdots \sum_i \frac{1}{2}\mu \frac{\partial}{\partial\mu}\delta_{z_i} = 0$$

From C-S, $\beta(g) = \mu \frac{\partial}{\partial\mu}\left(-\delta_g + \frac{1}{2} g \sum \delta_{z_i}\right)$

Associating terms, identify $\delta_g = -\beta \log \frac{\Lambda^2}{\mu^2}$ + finite

$\beta(g) = -2B - g\sum_i A_i$ to lowest order.

Interpretation: $\beta(g)$ & γ: independent of finite parts of counterterms. Any momentum scale μ^2 chosen for ren. cond. gives same results. Only coefficients of divergences arise.