Lecture 6.
May 11, 2020
QFT II.
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Last time: Renormalization group and ϕ^4 theory.
Callan-Symanzik equation.

Today: Continue Callan-Symanzik equation.

Save for next time [Understand β-fcn and anomalous dimension.
Asymptotic behavior in UV + IR.

Recall the $\lambda\phi^4$ example.

We set the renormalization condition $\Big($  $\Big)_{amp} = -i\lambda$

at scale $s = 4m^2$, $t=0$, $u=0$. Resulting matrix element was manifestly finite, since counterterm absorbed the $\left(\frac{2}{\epsilon}\right)$ divergences from dim. reg.

$$i\delta_\lambda = \frac{i\lambda^2}{32\pi^2}\int_0^1 dx \left(\frac{6}{\epsilon} - \log\Delta_s - \log\Delta_t - \log\Delta_u - 3\gamma + 3\log 4\pi \right)$$

with $\Delta_v = x^2 v - xv + m^2$

Consider using a diff. renorm. condition, for example, set $s = t = u = -\mu^2$, and neglect m^2.

We get
$$i\delta_\lambda = \frac{i3\lambda^2}{32\pi^2}\int_0^1 dx \left(\frac{2}{\epsilon} - \log\left(-x^2\mu^2 + x\mu^2\right) - \gamma + \log 4\pi \right)$$

$$= i\frac{3\lambda^2}{32\pi^2}\left(\frac{2}{\epsilon} - \log(\mu^2) + \text{finite} \right)$$

no μ dependence

This choice of scale μ is arbitrary, should not affect physical observables.

In fact, for two experimenters studying the same physical system, they may well choose μ^2 differently; but the physics must match smoothly onto each other as their different choices of μ^2 converge.

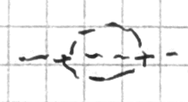
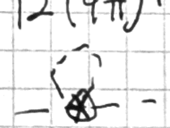
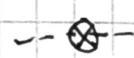
We get, since $iG^{(4)} \supset -i\delta_\lambda$,

$$\mu \frac{\partial G^{(4)}}{\partial \mu} = -\left(\frac{3\lambda^2}{32\pi^2} (\mu) \cdot \frac{-1}{\mu^2} \cdot 2\mu \right) = +\frac{3\lambda^3}{16\pi^2}$$

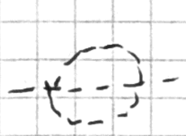
From Callen-Symanzik equation, this is the sum of the anomalous dimension term and the β-fcn term. We will claim that the anomalous dimension term has no term at $O(\lambda)$, and instead starts at $O(\lambda^2)$. Then, since the anomalous dimn. multiplies the renormalized $G^{(4)}$, the overall term in the C-S eqn. is $O(\lambda^3)$. By power counting, the $\frac{3\lambda^2}{16\pi^2}$ must be the β-fcn. term.

$$\beta(\lambda) = \frac{3\lambda^2}{16\pi^2} + O(\lambda^3).$$

Aside: Can calculate $\gamma(\lambda) = \frac{\lambda^2}{12(4\pi)^4}$ from

the two-loop  +  + 

$i\delta_\lambda$ $\qquad$ $i(p^2\delta_Z - \delta_m)$

where  $= -ip^2 \frac{\lambda^2}{12(4\pi)^4} \left(\frac{-1}{\epsilon} + \log p^2 + \dots \right)$

Here, the fact that $\gamma(\lambda)$ starts at $O(\lambda^2)$ is more of an exception than a rule.

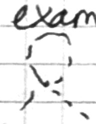
We can study the meaning of β + γ more abstractly by using ansätze for ~~the~~ particular Green's fcns.

Ref. Peskin + Schroeder Section 12.2, p. 413

In general, a massless 2 pt. Green's fcn. has

$$G^{(2)}(p) = \text{bare} + \text{loop} + \text{ct} + \dots$$

$$= \frac{i}{p^2} + \frac{i}{p^2}\left(A\log\frac{\Lambda^2}{-p^2} + \text{finite}\right) + \frac{i}{p^2}\left(p^2\delta_Z\right)\frac{i}{p^2} + \dots$$

(for example, )

Here, A is the coefficient of the divergence.

The C-S equation gives $\frac{-i}{p^2}\mu\frac{\partial}{\partial\mu}\delta_Z + 2\gamma\frac{i}{p^2} = 0$

$$\Rightarrow \gamma = \frac{1}{2}\mu\frac{\partial}{\partial\mu}\delta_Z$$

Explicitly, $\delta_Z = A\log\frac{\Lambda^2}{\mu^2}$ + finite, since it cancels the divergence and then $\gamma = -A$ to lowest order.

Furthermore,

$$G^{(n)} = \text{tree} + \text{1PI loop} + \text{vtx ct.} + \text{external leg.}$$

$$= -ig - iB\log\frac{\Lambda^2}{-p^2} - i\delta_g + (-ig)\sum_i\left(A_i\log\frac{\Lambda^2}{-p_i^2} - \delta_{Z_i} + \text{finite}\right)$$

C-S gives $\mu\frac{\partial}{\partial\mu}\left(\delta_g - g\sum\delta_{Z_i}\right) + \beta(g) + g\sum_i\frac{1}{2}\mu\frac{\partial}{\partial\mu}\delta_{Z_i} = 0$

$$\text{So, } \beta(g) = \mu\frac{\partial}{\partial\mu}\left(-\delta_g + \frac{1}{2}g\sum\delta_{Z_i}\right)$$

We know δ_g cancels the divergence of the vertex, so $\delta_g = -B\log\frac{\Lambda^2}{\mu^2}$ + finite and δ_{Z_i} cancels $G^{(2)}$ divergences.

$\Rightarrow \beta(g) = -2B - g\sum_i A_i$ to lowest order

Independent of finite parts.

Recall QED renormalization.

P+S, Sec. 10.3, p.332.

δ_1 :  $-ie\gamma^\mu\delta_1$

δ_2 :  $i(\not{p}\,\delta_2-\delta_m)$

δ_3 :  $-i(g^{\mu\nu}q^2-q^\mu q^\nu)\delta_3$

p.416 We found $\delta_1=\delta_2=\dfrac{-e^2}{(4\pi)^2}\dfrac{\Gamma(2-\frac{d}{2})}{(\mu^2)^{2-d/2}}$ + fin.

$$\delta_3=\frac{-e^2}{(4\pi)^2}\,\frac{4}{3}\,\frac{\Gamma(2-d/2)}{(\mu^2)^{2-d/2}} + \text{fin.}$$

Since $\beta(e)=\mu\dfrac{\partial}{\partial\mu}\left(-e\delta_1+e\delta_2+\dfrac{e}{2}\delta_3\right)$

$$=\mu\frac{\partial}{\partial\mu}\left(\frac{e}{2}\delta_3\right)=\frac{e^3}{12\pi^2}$$

Also, $\gamma_2(e)=\dfrac{e^2}{16\pi^2}$ ← electron propagator

$\gamma_3(e)=\dfrac{e^2}{12\pi^2}$ ← photon propagator