Lecture 5, May 6, 2020

Wednesday, May 6, 2020 2:06 PM

Last time: Refresher on QED renormalization
Ward identity discussion & connection between vertex & electron field renormalization in QED

Today: Continue renormalization group
Callan-Symanzik equation
Main calculation example: $\lambda\phi^4$ theory.

Consider $\lambda\phi^4$ theory:

bare $\mathcal{L} = \frac{1}{2}(\partial_\mu\phi)^2 - \frac{1}{2}m_0^2\phi^2 - \frac{\lambda_0}{4!}\phi^4$

We adopt wavefcn. rescaling $\phi = Z^{1/2}\phi_R$, ϕ is the bare field and ϕ_R is renormalized field.

$$\mathcal{L} = \frac{1}{2}Z(\partial_\mu\phi_R)^2 - \frac{1}{2}m_0^2 Z\phi_R^2 - \frac{\lambda_0}{4!}Z^2\phi_R^4$$

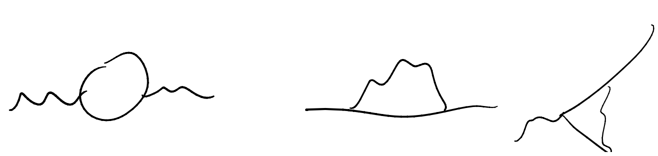
Split $Z = 1 + \delta_Z$ where δ_Z is the counterterm.

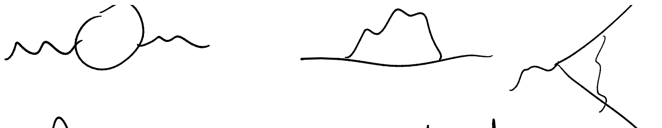
$$\delta_m = m_0^2 Z - m_R^2$$
$$\delta_\lambda = \lambda_0 Z^2 - \lambda$$

renorm. $\mathcal{L} = \frac{1}{2}(\partial_\mu\phi_R)^2 - \frac{1}{2}m_R^2\phi_R^2 - \frac{\lambda}{4!}\phi_R^4$ ← exactly as original, but all fields + couplings are normalized

$$+ \frac{1}{2}\delta_Z(\partial_\mu\phi_R)^2 - \frac{1}{2}\delta_m\phi_R^2 - \frac{\delta_\lambda}{4!}\phi_R^4$$

Recall that δ's are formally infinite, then this is the "offset" to move from the "infinity" of the $\mathcal{L}_{bare}$ local description to the perturbative $\mathcal{L}_{renormalized}$.



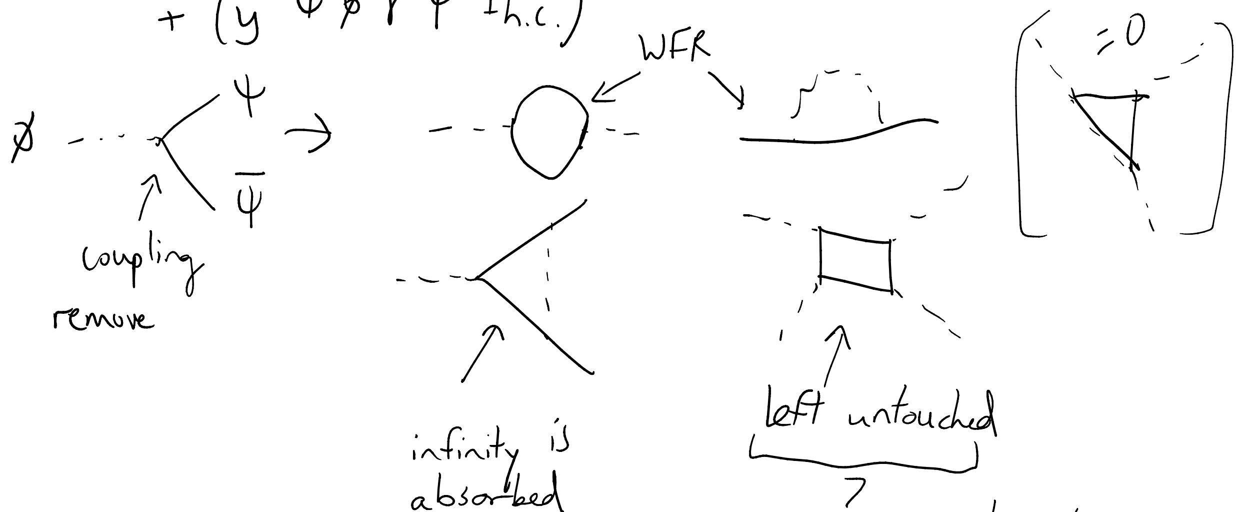


Divergences are matched in phys. ...

One-loop diagrams are built from tree-level interactions

Pseudoscalar Yukawa

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 + \bar{\Psi} i\not{\partial} \Psi - m_f \bar{\Psi}\Psi - \frac{1}{2} m_3^2 \phi^2$$
$$+ \left(y \bar{\Psi} \phi \gamma^5 \Psi + h.c.\right)$$

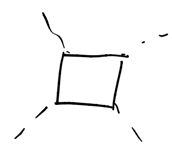


$$D = \int \frac{d^4k}{(2\pi)^4} \left(\frac{k}{k^2}\right)^4 = 0$$

log divergent

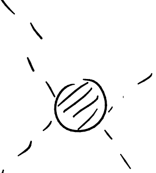
$\mathcal{L} \mathrel{+}= \lambda \phi^4$

[tree-level $= 0$]

at one-loop, regenerated by 

proportional to y^4.

Tree-level bare coupling [dimn-less] still holds information. At the fundamental level, tree-level $= 0$ is not a special choice. If tree-level $= \#$, , counterterm necessary.

One-loop ren. cond. applies to all $\mathcal{O}(y^4)$ and $\mathcal{O}(\lambda^2)$ contributions to , eliminating

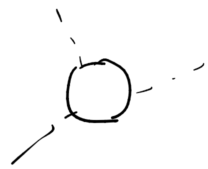
$\mathcal{O}(\lambda^2)$ does not permit you to eliminate counterterm.

"Special choice":

[illegible]

"Special choice":

Distinction between sending a parameter $\to 0$ + whether you recover a symmetry.

Recall  vanishes. There is not $\lambda_3 \phi^3$ at tree-level.

At one-loop, diagram exists, but $\mathcal{L} = y \phi \bar{\Psi} \gamma^5 \Psi$ + h.c.

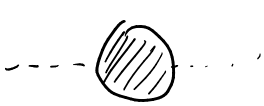
Parity symmetry $= \mathbb{Z}_2$ symmetry [Assign parity where $\phi \sim 1$, $\Psi_L \sim \frac{1}{2}$, $\Psi_R \sim \frac{-1}{2}$ (axial symmetry, Ψ_L + Ψ_R transform oppositely). The ϕ^3 is not invariant. Parity symmetry is respected in all one-loop + multiloop diagrams. So $\lambda_3 \phi^3$ term does not respect symmetry + is vanishing from $\mathcal{O}(y^3)$.

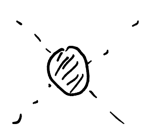
When we set $\underbrace{\lambda \phi^4}$ $\lambda \to 0$ (will fix charges)

Tree-level symmetries of Lagrangian control the loop-level operators + corrections. [Exception: quantum anomalies are when tree-level symmetries are broken by quantum effects.]

Infinite offset (from counterterms) are defined by renormalization conditions.

Renormalization conditions for $\lambda \phi^4$:

 $= \frac{i}{p^2 - m_R^2}$ + non-singular at $p^2 = m_R^2$

$\Big($  $\Big)_{\text{amputated}} = -i\lambda$ at $s = 4m^2$, $t = u = 0$.

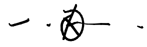
Feynman rules: (from second Lagrangian)

 $\frac{i}{p^2 - m^2}$.  $-i\lambda$

Feynman rules

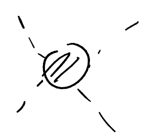
$\frac{i}{p^2-m^2+i\epsilon}$

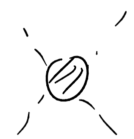
$-i\lambda$



$i(p^2\delta_Z-\delta_m)$

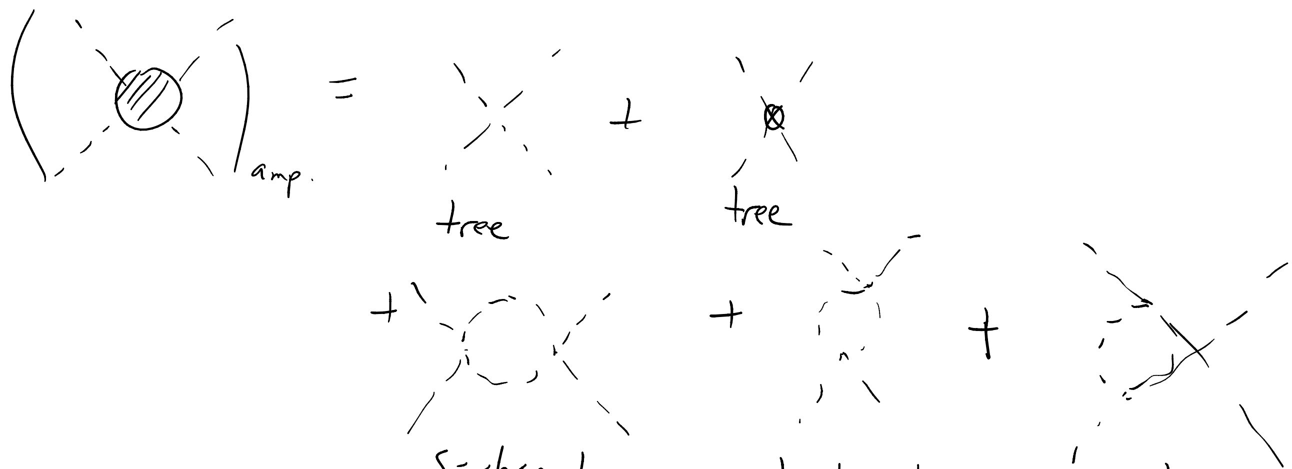
$-i\delta_\lambda$

Calculate + impose $-i\lambda = \Big($  $\Big)_{\text{amputated}}$ @ 1loop.

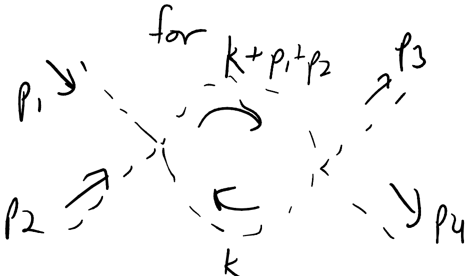
Exercise: draw all tree- + one-loop diagrams for 

Back @ 3:10pm.

$$\mathcal{L} = \tfrac{1}{2}(\partial_\mu\phi_R)^2 - \tfrac{1}{2}m_R^2\phi_R^2 - \frac{\lambda}{4!}\phi_R^4 + \tfrac{1}{2}\delta_Z(\partial_\mu\phi_R)^2 - \tfrac{1}{2}\delta_m\phi_R^2 - \frac{\delta_\lambda}{4!}\phi_R^4$$



Matrix element for



[Lazy with μ^{4-d} for energy scaling + $i\epsilon$ in propagators.]

$$i\mathcal{M} = \int\frac{d^4k}{(2\pi)^4}\,\frac{(-i\lambda)^2}{2}\,\frac{i}{k^2-m^2}\,\frac{i}{(k+p_1+p_2)^2-m^2}$$

① Combine denominators using Feynman parameters.

$$= \frac{\lambda^2}{2}\int\frac{d^4k}{(2\pi)^4}\int dx\,dy\,\frac{\delta(x+y-1)}{\quad}$$

$$= \frac{\lambda^3}{2} \int \frac{d^4k}{(2\pi)^4} \int dx\,dy \frac{\delta(x+y-1)}{\left(k^2 - m^2 + x(2k)\cdot(p_1+p_2) + x(p_1+p_2)^2\right)^2}$$

$$= \frac{\lambda^2}{2} \int \frac{d^4k}{(2\pi)^4} \int dx \frac{1}{\left(k^2 + x\,2k\cdot(p_1+p_2) + x(p_1+p_2)^2 - m^2\right)^2}$$

② Shift loop momenta & complete square

$$l \equiv k + x(p_1+p_2)$$

$$i\mathcal{M} = \frac{\lambda^2}{2} \int \frac{d^4l}{(2\pi)^4} \int dx \frac{1}{\left(l^2 + x(p_1+p_2)^2 - m^2 - x^2(p_1+p_2)^2\right)^2}$$

$$\Delta \equiv x^2(p_1+p_2)^2 - x(p_1+p_2)^2 + m^2$$

Mandelstam $s = (p_1+p_2)^2$

$$\Delta \equiv x(x-1)s + m^2$$

③ Use dim. reg.

$$i\mathcal{M} = \frac{\lambda^2}{2} \int \frac{d^dl}{(2\pi)^d} \int dx \frac{1}{(l^2-\Delta)^2}$$

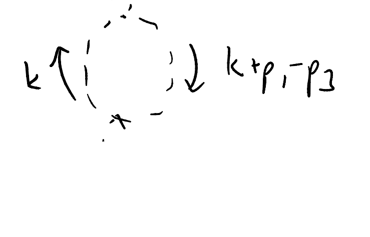
Formula: $$\int \frac{d^dl}{(2\pi)^d} \frac{1}{(l^2-\Delta)^n} = \frac{(-1)^n\, i}{(4\pi)^{d/2}} \frac{\Gamma(n-\frac{d}{2})}{\Gamma(n)} \left(\frac{1}{\Delta}\right)^{n-\frac{d}{2}}$$

And $d = 4-\epsilon$ gives

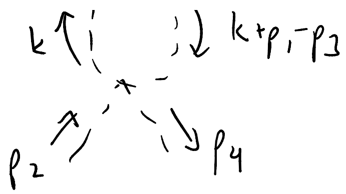
$$\frac{\Gamma(2-\frac{d}{2})}{(4\pi)^{d/2}} \left(\frac{1}{\Delta}\right)^{2-\frac{d}{2}} = \frac{1}{(4\pi)^2}\left(\frac{2}{\epsilon} - \log\Delta - \gamma + \log 4\pi + \mathcal{O}(\epsilon)\right)$$

$$i\mathcal{M}_s = \frac{\lambda^2}{2} \int dx \cdot \frac{1}{(4\pi)^2} \cdot \frac{i \cdot 1}{\Gamma(2)} \left(\frac{2}{\epsilon} - \log\Delta - \gamma + \log 4\pi + \mathcal{O}(\epsilon)\right)$$

$$i\mathcal{M}_s = \frac{\lambda^2}{2} \left(\frac{i}{16\pi^2}\right) \int dx \left(\frac{2}{\epsilon} - \log\Delta_s - \gamma + \log 4\pi + \mathcal{O}(\epsilon)\right)$$



Similarly, gives $\Delta_t = x^2t - xt + m^2$



gives $\Delta_t = x^2 t - xt + m^2$

& $\Delta_u = x^2 u - xu + m^2$

$$i\mathcal{M}_{tot} = -i\lambda + -i\delta_\lambda + f(s) + f(t) + f(u)$$

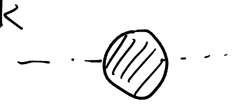
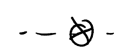
Require $f(s) + f(t) + f(u) = i\delta_\lambda$ at $s = 4m^2,\ t = u = 0$
for renorm. cond.

Explicitly, $$i\delta_\lambda = \frac{i\lambda^2}{32\pi^2}\int_0^1 dx\left(\frac{6}{\epsilon} - 3\gamma + 3\log 4\pi - \log\Delta_s\Big|_{4m^2} - 2\log m^2\right)$$

$$i\mathcal{M} = -i\lambda + \frac{i\lambda^2}{32\pi^2}\int_0^1 dx \log\left(\frac{x^2 s - xs + m^2}{4x^2m^2 - 4xm^2 + m^2}\right)$$
$$+ \frac{i\lambda^2}{32\pi^2}\int_0^1 dx\left[\log\left(\frac{x^2 t - xt + m^2}{m^2}\right) + \log\left(\frac{x^2 u - xu + m^2}{m^2}\right)\right]$$

$i\delta_\lambda$ removes $\frac{6}{\epsilon}$ and finite contributions.
Replaced by $\log\Delta_s\big|_{s=4m^2}$, $\log\Delta_t\big|_{t=0}$, $\log\Delta_u\big|_{u=0}$

Nothing divergent as $\epsilon \to 0$.

Can check  $= \dfrac{i}{p^2 - m^2 - M^2(p^2)}$
with expansion at one-loop from  and 

Imposing $M^2(p^2)\big|_{p^2=m^2} = 0$ and $\frac{d}{dp^2}M^2(p^2)\Big|_{p^2=m^2} = 0$

$\Rightarrow \delta_Z = 0$ $\qquad \delta_m = \dfrac{-\lambda}{2(4\pi)^{d/2}}\dfrac{\Gamma(1-\frac{d}{2})}{(m^2)^{1-d/2}}$

Pivot back to the renormalization group.
Consider defining the renormalization conditions differently.

Pivot back to the renormalization group.

Consider defining the renormalization conditions differently.

For convenience, define at scale μ:

Relation between bare + renormalized Green's fcns. is

$$\langle \Omega | T\phi(x_1)\ldots\phi(x_n)|\Omega\rangle = Z^{-n/2}\langle \Omega | T\phi_0(x_1)\ldots\phi_0(x_n)|\Omega\rangle$$

Differentiate bare Green's fcn. : no dependence on μ.

$$\mu\frac{d}{d\mu}G_n^{(0)} = 0 = \underbrace{\mu\frac{d}{d\mu}\left(Z^{n/2}G_n\right)}_{\text{full chain rule deriv.}}$$

$$\left[\ dG_n = \frac{\partial G}{\partial x}dx + \frac{\partial G}{\partial \lambda}d\lambda + \frac{\partial G}{\partial \mu}d\mu\right.$$

$$= \mu\,\frac{n}{2}\,\frac{Z^{n/2}}{Z}\left(\frac{dZ}{d\mu}\right)G_n + \mu Z^{n/2}\frac{\partial G_n}{\partial \mu}$$

$$+ \mu Z^{n/2}\left(\frac{\partial \lambda}{\partial \mu}\right)\frac{\partial G_n}{\partial \lambda} \quad \text{for massless } \phi^4.$$

$$\Rightarrow 0 = \left[n\left(\frac{1}{2}\frac{\mu}{Z}\frac{\partial Z}{\partial \mu}\right) + \mu\frac{\partial \lambda}{\partial \mu}\frac{\partial}{\partial \lambda} + \mu\frac{\partial}{\partial \mu}\right]G$$

Callen-Symanzik equation.

Discusses the variation of renormalized Green's fcn. wrt. scale μ, which defines renormalization conditions.

Intuitively, have to recover same theory if renorm. conditions / measurements at diff. scales "agree"

Most important coefficients in Callen-Symanzik eqn:

$$\beta(\lambda) = \mu\frac{\partial\lambda}{\partial\mu} = \frac{\partial\lambda}{\partial\log\mu} \quad \text{"}\beta\text{- fcn. of coupling } \lambda\text{"}$$

Change in coupling strength as you vary scale μ.

$$\gamma(\lambda) = \frac{1}{2}\frac{\mu}{Z}\frac{\partial Z}{\partial\mu} \quad \text{"anomalous dimension"}$$

$$\gamma(\lambda) = \frac{1}{2}\frac{\mu}{Z}\frac{\partial Z}{\partial \mu}$$ "anomalous dimension"

Deviation from classical scaling behavior of G.

Include masses:

$$\left(\mu\frac{\partial}{\partial\mu} + \beta(\lambda)\frac{\partial}{\partial\lambda} + n\gamma(\lambda) + \gamma_{\phi^2}\, m^2\frac{\partial}{\partial m^2}\right) G^{(n)}(\{p_i\}, \mu, \lambda, m^2) = 0$$

$\gamma_{\phi^2} = \mu\frac{\partial}{\partial\mu}\log Z_{\phi^2}(\mu)$, where operator $\phi^2_{bare} = Z_{\phi^2}\phi^2_r$