Lecture 4, April 29, 2020

Wednesday, April 29, 2020 1:08 PM

Last time:

super-renormalizeable, renormalizable, + non-renormalizable theories

Counterterms + superficial degree of divergence

Optical theorem.

Today: Refresher on QED renormalization

Ward identity + connection / protection between vertex + electron field renorm.

The renormalization group + renormalization flow

Callen-Symanzik equation

Reminder: Homework 1 due on May 4 @ 9:15 am to yu001@uni-mainz.de

Monday discussion from 9:15-11:00 am.

QED:

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu}F^{\mu\nu} + \bar{\Psi}\left(i\partial\!\!\!/ - m - e\gamma^\mu A_\mu\right)\Psi$$

$$\left[+\ \bar{\Psi}\left(iD\!\!\!\!/ - m\right)\Psi\right]$$

$$D\!\!\!\!/ = \partial\!\!\!/ + ie\gamma^\mu A_\mu$$

4 renormalization conditions:

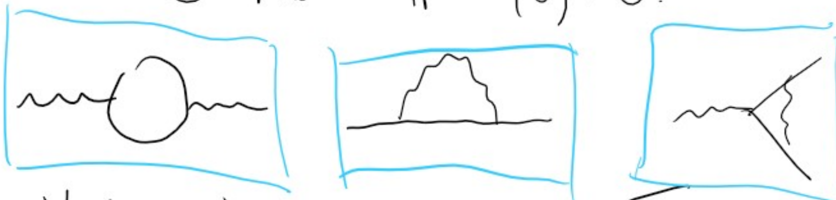
① Pole location $\Pi(m_p) = 0$] fermion

② Pole residue $\Pi'(m_p) = 0$

③ $\Gamma^\mu(0) = \gamma^\mu$ vertex

④ Photon $\Pi^{\mu\nu}(0) = 0$.

Given prescription of adopting counterterms, how are these infinities related between different diagrams?



Vertex has $Z_1 = Z_e Z_2 \sqrt{Z_3}$

Z_1: vertex; Z_e: charge; Z_2: electron renorm.; $\sqrt{Z_3}$: photon

Found $\boxed{Z_1 = Z_2}$ to all orders in pert. theory.

General result

found $\boxed{Z_1 = Z_2}$ to all orders in pert. theory.

General result of Ward-Takahashi identity.

Vertex & electron field renorm. are matched, so $\boxed{\bar\Psi i\partial\!\!\!/ \Psi}$ & $\boxed{\bar\Psi e_R \gamma^\mu A_\mu \Psi}$ do not develop a relative factor at any order in perturbation theory.

IOW, this combination $\boxed{\bar\Psi i D\!\!\!\!/ \Psi}$ does not recieve radiative corrections.

Why is it important that $Z_1 = Z_2$?

Consider quark & electron:

$$\mathcal{L} \mathrel{+}= i\bar\Psi_q \left(\partial\!\!\!/ + i\tfrac{2}{3} e_R A\!\!\!/ \right) \Psi_q - m_q \bar\Psi \Psi$$

Ratio of charges between quark & electron is 2/3 in bare theory & kept protected in renormalized $\Rightarrow$ no corrections.

Quark field renormalization would necessarily gluons.

In fact, this identity between the infinities of the matter field renormalization & the vertex is known as the Ward-Takahashi identity.

Sketch of Proof: Weinberg (p. 447). (Will have $-+++$ metric convention)

Take vertex fcn.

$$\int d^4x\, d^4y\, d^4z\; e^{-ip\cdot x} e^{-ik\cdot y} e^{+il\cdot z} \langle \Omega | T\{ J^\mu(x) \Psi_n(y) \bar\Psi_m(z) \} | \Omega \rangle$$

$$\equiv -iq\, D_{nn'}(k)\, \Gamma^\mu_{n'm'}(k,l)\, D_{m'm}(l)\, \delta^{(4)}(p+k-l)$$

$D_{nn'}$ is complete Dirac propagator.

Γ^μ is sum of vertex graphs w/ one incoming Dirac line, one outgoing Dirac line, & one photon line, all amputated.

Use identity

$$\star = \frac{\partial}{\partial x^\mu} T\{ J^\mu(x) \Psi_n(y) \bar\Psi_m(z) \}$$

(See p.447)

$$= T\{ \partial_\mu J^\mu(x) \Psi_n(y) \bar{\Psi}_m(z)$$
$$+ \delta(x^0 - y^0)\, T\{ [J^0(x), \Psi(y)]\, \bar{\Psi}_m(z) \}$$
$$+ \delta(x^0 - z^0)\, T\{ \Psi_n(y)\, [J^0(x), \bar{\Psi}_m(z)] \} \}$$

$\partial_\mu J^\mu(x) = 0$ by conservation

$$[J^0(x), \Psi_\ell(y)] = -q_\ell \Psi_\ell(y)\, \delta^{(3)}(\vec{x} - \vec{y})$$
$$[J^0(x), \bar{\Psi}_\ell(y)] = q_\ell \bar{\Psi}_\ell(y)\, \delta^{(3)}(\vec{x} - \vec{y})$$

We get

$$\star = -q\, \delta^4(x-y)\, T\{ \Psi_n(y) \bar{\Psi}_m(z) \}$$
$$+ q\, \delta^4(x-z)\, T\{ \Psi_n(y) \bar{\Psi}_m(z) \}$$

Fourier Transform:

$$\Rightarrow (\ell - k)_\mu\, D(k)\, \Gamma^\mu(k,\ell)\, D(\ell) = i D(\ell) - i D(k)$$

or $$(\ell - k)_\mu \Gamma^\mu(k,\ell) = \frac{i}{D(k)} - \frac{i}{D(\ell)}$$

Ward-Takahashi identity.

Special case $\boxed{\ell \to k} \Rightarrow \quad \Gamma^\mu(k,k) = -i \frac{\partial}{\partial k_\mu} D^{-1}(k)$

Recall $D^{-1}(k) = i\not{k} - m + \Pi^*(\not{k})$

$$\Gamma^\mu(k,k) = \gamma^\mu + i \frac{\partial}{\partial k_\mu} \Pi^*(\not{k})$$

On mass shell:

$$\bar{u}\, \Gamma^\mu(k,k)\, u = \bar{u}_k \gamma^\mu u \quad \text{with} \quad i(\not{k} - m) u = 0.$$

Equivalently $\Gamma^\mu(p+k, p) \to Z_1^{-1} \gamma^\mu$ for $k \to 0$.

Recall $S(p) \sim \frac{i Z_2}{\not{p} - m}$ for Z_2 as residue of pole.

Combine: $k \to \ell + k$, $\quad k_\mu \Gamma^\mu(\ell+k, \ell) = i D^{-1}(\ell+k) - i D^{-1}(\ell)$

$$Z_1^{-1} \not{k} \qquad\qquad = i\, \frac{Z_2^{-1}}{i} \not{k}$$

See $\boxed{Z_1 = Z_2}$

See $\boxed{Z_1 = Z_2}$

Interpretation

Current conservation on 3-particle amplitude.

On one side, as particles get closer in momentum, we extract a pure γ^μ vertex and Z_1 factor.

On other side, matching on-shell degrees of freedom, we connect to the residue of fermion propagators, give Z_2.

Comments:

Gauge invariance: fundamental symmetry of Lagrangian.

Current conservation: EOM following symmetry transformation.

Ward identity: diagrammatic identity that imposes symmetry structure on amplitudes.

Regulators can violate WI + gauge invariance — exception is dim. reg.

Fact that symmetries control renormalization is main focus now.

Break for 10 min + return @ 3:17pm.

Introduce the renormalization group.

Remind you of two popular methods for determining counterterms.

On-shell scheme vs. minimal subtraction.

In general, counterterm is a formally infinite # that acts in $\mathcal{L}$ as a new Feynman rule for removing UV divergences.

[Recall: $Z = 1 + \delta$, δ acts as a feynman rule.]

Minimal subtraction: no finite part, only remove infinity.

Modified minimal subtraction: in dim. reg, infinity (are $\frac{1}{\epsilon}$, $d = 4-\epsilon$) also come with $\log 4\pi$ + γ_E, these are also removed.

On-shell scheme: Shifts renormalized finite constants to on-shell values like $m_R \to m_p$. pole mass.
(See Schwartz, Chap. 18.)

Motivation for extracting the essence of renormalization + interpreting the physics of infinities.

Two viewpoints on the renormalization group; i.e. observables are independent of changes in the way they are calculated.

1. Wilsonian: In a finite theory w/ UV cutoff Λ, physics at $E \ll \Lambda$ is independent of precise Λ value. Changing Λ changes couplings in theory such that observables remain same.

2. Continuum: Observables are independent of renormalization conditions, in particular, of the scale where we define renormalized quantities. Invariance remains after theory is renormalized and we remove cutoff ($\Lambda \to \infty$, $d=4$)
In dim. reg. with $\overline{MS}$, scales are replaced by $\tilde{\mu}$, RG (ren. group) comes from $\tilde{\mu}$ independence.

Consider renormalization of $\lambda\phi^4$.

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi)^2 - \frac{1}{2}m_0^2\phi^2 - \frac{\lambda_0}{4!}\phi^4$$

Use wavefcn. rescaling $\phi = Z^{1/2}\phi_R$

$$\mathcal{L} = \frac{1}{2}Z(\partial_\mu\phi_R)^2 - \frac{1}{2}m_0^2 Z\phi_R^2 - \frac{\lambda_0}{4!}Z^2\phi_R^4$$

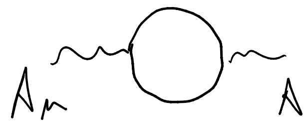
Define: $Z = 1+\delta_Z$. $\delta_m = m_0^2 Z - m^2$ $\delta_\lambda = \lambda_0 Z^2 - \lambda$.

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi_R)^2 - \frac{1}{2}m^2\phi_R^2 - \frac{\lambda}{4!}\phi_R^4$$
$$+\frac{1}{2}\delta_Z(\partial_\mu\phi_R)^2 - \frac{1}{2}\delta_m\phi_R^2 - \frac{\delta_\lambda}{4!}\phi_R^4$$

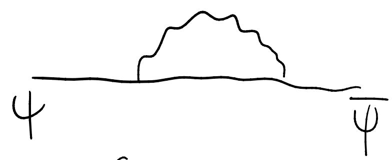
Go back to QED.

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu}F^{\mu\nu} + i\bar{\Psi}\not{\partial}\Psi - m\bar{\Psi}\Psi$$

2-pt. correlation fcns.



$$\int d^4k \left(\frac{\not{k}}{k^2}\right)^2 \Rightarrow D=2.$$

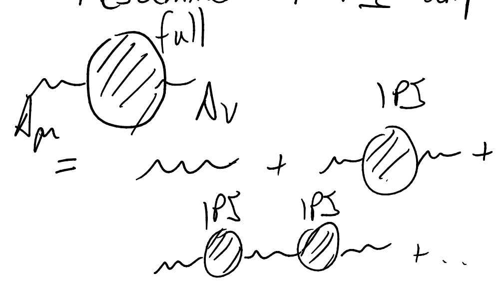


$$\int d^4k \frac{\not{k}}{k^2}\cdot\frac{1}{k^2} \Rightarrow D=1$$

Usual trick/algorithm/procedure

Full 2-pt. corr. fcn.

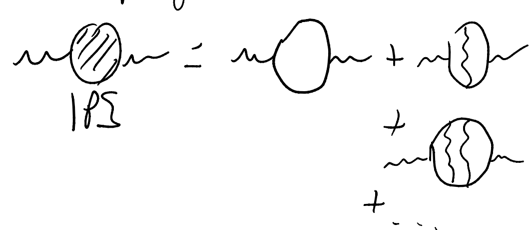
resummed 1-PI amplitude



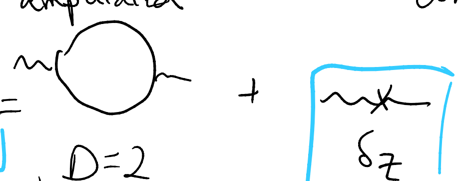
$$\Pi_{\mu\nu} = \frac{-ig_{\mu\nu}}{p^2} + \frac{-ig_{\mu\nu}}{p^2}\Pi^{1PI}_{\nu\rho}(p^2)\frac{-ig_{\rho\sigma}}{p^2} + \dots$$

$$= \frac{-ig_{\mu\nu}}{p^2 - \underbrace{\Pi^{1PI}(p^2)}_{\text{amputated two-pt. correlation fcn.}}}$$

Perturbative exp. of 1PI of couplings



Π^{1PI} at $\mathcal{O}(e^2)$ =



sum is finite expression for all p^2.

Counter-term corresponds to new $\mathcal{L}$ term

$$\mathcal{L} \mathrel{+}= \frac{\delta_Z}{4}\left[A_\mu A_\nu\right]$$

combination is the $\boxed{p^2\to 0}$ of Π^{1PI} at $\mathcal{O}(e^2)$

$$-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} \Rightarrow -\frac{1}{4}\overbrace{\left(\underset{\text{bare}}{1}+\underset{\text{infinite}}{\delta_Z}\right)}F_{\mu\nu}F^{\mu\nu} = -\frac{1}{4}Z F_{\mu\nu}F^{\mu\nu}$$

$$-\frac{1}{4} F_{\mu\nu}F^{\mu\nu} \to -\frac{1}{4}(1+\delta_Z)F_{\mu\nu}F^{\mu\nu} = -\frac{1}{4} Z F_{\mu\nu}F^{\mu\nu}$$

bare

infinity cancelling counterterm

Every bare term in $\mathcal{L}$ (all derivatives + masses + couplings) have a separation into formally infinite piece, which is made explicit by counterterm + the remaining finite piece, which only deals with renormalized fields + couplings.

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi)^2 - \frac{1}{2}m_0^2\phi^2 - \frac{\lambda_0}{4!}\phi^4$$ ← bare Lagrangian

Use wavefcn. rescaling $\phi = Z^{1/2}\phi_R$

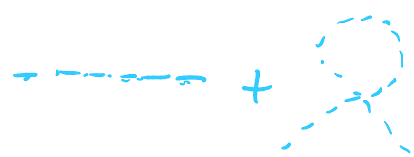
$$\mathcal{L} = \frac{1}{2}Z(\partial_\mu\phi_R)^2 - \frac{1}{2}m_0^2 Z\phi_R^2 - \frac{\lambda_0}{4!}Z^2\phi_R^4$$

Define: $Z = \boxed{1}+\boxed{\delta_Z}$. $\boxed{\delta_m} = \boxed{m_0^2 Z} - \boxed{m^2}$ $\quad \delta_\lambda = \boxed{\lambda_0} Z^2 - \lambda$.

$$\mathcal{L} = \boxed{\frac{1}{2}(\partial_\mu\phi_R)^2 - \frac{1}{2}m^2\phi_R^2 - \frac{\lambda}{4!}\phi_R^4}$$ → renormalized $\mathcal{L}$.

$$+\frac{1}{2}\delta_Z(\partial_\mu\phi_R)^2 - \frac{1}{2}\delta_m\phi_R^2 - \frac{\delta_\lambda}{4!}\phi_R^4$$ ← explicit counterterm vertices that will cancel divergences

$\phi\phi$:



$+ \mathcal{O}(\lambda^2)$

$\mathcal{O}(\lambda)$

$D=2$

Question: Why can you just add new terms (w/ infinite coefficients)?

Should think of $\mathcal{L}$ as <u>local</u> description + parameters as <u>family</u> of parameters.