QFT

Lecture 4.

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Last time: Super-renormalizable, Renormalizable, Non-renormalizable
Counterterms + degree of divergence
Optical theorem.

Today: Refresher on QED renormalization
Ward identity + Connection between vertex + electron field renorm.

Next time: [The renormalization group.]
[Callen-Symanzik equation]

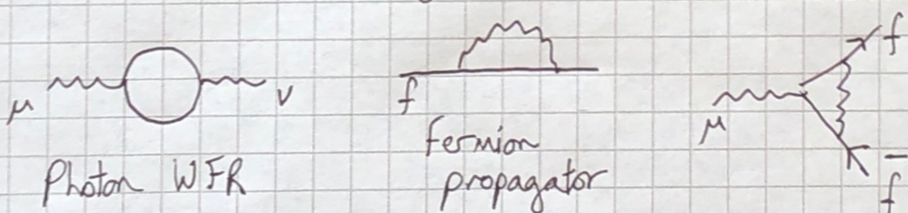
We will now continue our study of interacting field theories and the discussion of renormalization by presenting a few example theories.

First, recall QED.

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu}F^{\mu\nu} + \bar{\Psi}(i\not{D} - m)\Psi$$

for $\not{D} = \not{\partial} - ie\not{A}$, since Ψ has charge -1.

We had the following divergent diagrams at one-loop.



Photon WFR (wavefcn. renormalization) | Fermion propagator | Vertex

We had 4 renormalization conditions:

1. Pole location of fermion propagator corr. $\Pi_f(m_R) = 0$
2. Pole residue of fermion propagator corr. $\Pi_f'(m_R) = 0$
3. Vertex at $q^2 \to 0$ $\Gamma^\mu(q^2=0) = \gamma^\mu$
4. $\Pi_{photon}(0) = 0$

(2)

Vertex has $Z_1 \equiv Z_e \; Z_2 \; \sqrt{Z_3}$

Vertex — charge — Electron field renorm. — photon field

Found $Z_1 = Z_2$ to all orders in pert. theory

In other words, vertex with photon divergence matches the electron field renormalization divergence. This is not obvious from the fact that the diagrams seem independent, but it can be understood in the structure of the way the covariant derivative is ~~cont~~ constructed.

As a consequence, $\bar{\Psi} i \not{\partial} \Psi$ and $\bar{\Psi} e_R \gamma^\mu A_\mu \Psi$ do not receive a relative factor in radiative corrections.

If they did, though, would be disastrous. Consider adding a quark: $i\bar{\Psi}_q (\not{\partial} + i\frac{2}{3} e_R \not{A}) \Psi_q - m_q \bar{\Psi}\Psi$. If there were a relatively factor, then the ratio of electron & quark charges would not be protected, & neutrality of atoms would not be guaranteed. This does not occur, & the ratio is fixed, even though quarks have a more complicated WFR.

This is a general result of the Ward-Takahashi identity.

Can prove the Ward-Takahashi identity.

p. 446, Section 10.4 Following Weinberg, which uses $-+++$ metric convention. consider vertex fcn.

$$\int d^4x \, d^4y \, d^4z \; e^{-ip\cdot x} e^{-ik\cdot y} e^{+il\cdot z} \langle \Omega | T\{ J^\mu(x) \Psi_n(y) \bar{\Psi}_m(z) \} | \Omega \rangle$$
$$\equiv -iq \, D_{nn'}(k) \, \Gamma^\mu_{n'm'}(k,l) \, D_{m'm}(l) \, \delta^{(4)}(p+k-l)$$

for particle with charge q +

$-i D_{nm}(k) \, \delta^4(k-l)$ is the complete Dirac propagator.

Also, Γ^μ is sum of vertex graphs with one incoming Dirac line, one outgoing Dirac line, & one photon line, but all external states stripped away. Then $\Gamma^\mu(k,l) \to \gamma^\mu$ in the limit of no interactions and $D_{nm} \to \dfrac{1}{\not p - m}$ (+ - - - metric)

We derive relation between Γ^μ and D_{nm} by the identity

$$\frac{\partial}{\partial x^\mu} T\{ J^\mu(x) \Psi_n(y) \bar\Psi_m(z)\}$$

$$= T\{ \partial_\mu J^\mu(x) \Psi_n(y) \bar\Psi_m(z)\}$$

$$+ \delta(x^0 - y^0)\; T\{ [J^0(x), \Psi_n(y)]\, \bar\Psi_m(z)\}$$

$$+ \delta(x^0 - z^0)\; T\{ \Psi_n(y)\, [J^0(x), \bar\Psi_m(z)]\}$$

δ-fcns from time derivs. of step-fcns.

Have $\partial_\mu J^\mu = 0$ by current conservation

Use commutation relations

$$[J^0(x), \Psi_\ell(y)] = -q_\ell \Psi_\ell(y)\, \delta^3(\vec x - \vec y)$$

$$[J^0(x), \bar\Psi_\ell(y)] = q_\ell\, \bar\Psi_\ell(y)\, \delta^3(\vec x - \vec y)$$

Obtain $\dfrac{\partial}{\partial x^\mu} T\{ J^\mu(x) \Psi_n(y) \bar\Psi_m(z)\}$

$$= -q\, \delta^4(x-y)\, T\{\Psi_n(y) \bar\Psi_m(z)\}$$

$$+ q\, \delta^4(x-z)\, T\{\Psi_n(y) \bar\Psi_m(z)\}$$

Fourier transform

$$\Rightarrow \quad (l-k)_\mu D(k)\, \Gamma^\mu(k,l)\, D(l) = iD(l) - iD(k) \qquad \text{General identity}$$

$$\text{or} \quad (l-k)_\mu \Gamma^\mu(k,l) = iD^{-1}(k) - iD^{-1}(l) \quad \boxed{\checkmark}$$

The W-T identity is usually discussed in the special case

$l \to k$, then $\quad \Gamma^\mu(k,k) = -i \dfrac{\partial}{\partial k_\mu} D^{-1}(k)$

Since $D^{-1}(k) = i\not k - m + \Pi(\not k)$ (+ - - metric)

we have $\quad \Gamma^\mu(k,k) = \gamma^\mu + i \dfrac{\partial}{\partial k_\mu} \Pi(\not k)$

For renormalized Dirac field,

$$\bar u_k \Gamma^\mu(k,k)\, u_k = \bar u_k \gamma^\mu u_k, \text{ with } i(\not k - m)\, u(k) = 0$$

So, "renormalization of fermion field ensures radiative corrections to the vertex fcn. Γ_μ cancel when a fermion on mass shell interacts with an EM field with zero momentum transfer.

Equivalently, $\Gamma^\mu(p+k, p) \rightsquigarrow Z_1^{-1} \gamma^\mu$ for $k \to 0$

$$D(\not p) \sim \frac{i Z_2}{\not p - m} \quad \text{where } Z_2 \text{ is residue of pole.}$$

Using $\quad k_\mu \Gamma^\mu(p+k, p) = i D^{-1}(p+k) - i D^{-1}(\not p)$

$$Z_1^{-1} \not k = i \frac{Z_2^{-1}}{i} \not k$$

$$\Rightarrow Z_1 = Z_2 .$$

Discussion:

Gauge invariance is fundamental symmetry of Lagrangian.

Ward-Takahashi identity is also seen as diagrammatic identity that imposes symmetry on amplitudes. Can replace ext. pol. vector of photon in matrix element by 4-momentum + diagram should vanish. (Distinct from Furry's theorem.)

Regulators can violate Ward identity + gauge invariance, for example Pauli-Villars + hard momentum cutoff.

Dim. reg. keeps gauge invariance.

~~[illegible]~~

While Ward identity controls some family of ~~[illegible]~~ terms generated by renormalization, and symmetries more generally dictate the strength of new terms from renormalization, the general rule is that everything not forbidden by symmetry is <u>mandatory</u> + will be generated. Example: ϕ^4 term from cubic scalar Lagrangian.

Begin the renormalization group.

How do we define a physical model?

We have seen that infinities generally arise in interacting theories, but also that infinities in diff. diagrams can be matched, like in QED, as dictated by the Ward identity.

We recall that a simple construction of eliminating infinities is through the use of counter-terms. Given a bare Lagrangian, introducing counterterms leads to the definition of a renormalized Lagrangian. The counter-terms are fixed by a set of renormalization conditions, and renormalizeable theories only require a finite set of counterterms. Hence, renormalizeable theories, even though loop calculations are sometimes divergent, give finite physical predictions.

Recall there are two popular schemes for determining counterterms:

on-shell scheme vs. minimal subtraction.

On-shell: Shifts renormalized finite consts. to on-shell values like $m_R \to m_P$ (pole mass).

minimal subtraction: no finite part, only remove infinity

Modified minimal subtraction: in dim. reg., remove $\log 4\pi$ and γ_E.

Moving beyond masses + interactions, we consider the role of infinities + counterterms with regards to observables.

The renormalization group: invariance of observables under changes in way things are calculated.

Wilsonian: In a finite theory with UV cutoff Λ, physics at energies $E \ll \Lambda$ is independent of precise Λ value. Changing Λ changes couplings in theory s.t. observables remain same.

Continuum: Observables are independent of renormalization conditions, in particular the scales at which we defined renormalized quantities. Invariance remains after theory is renormalized + cutoff is removed ($\Lambda = \infty$, $d=4$). In dim. reg. with $\overline{MS}$, scales are replaced by μ, + RG comes from μ independence.