Lecture 3, April 27, 2020

Monday, April 27, 2020 8:09 AM

Last time:

- More on LSZ reduction formula
- Wavefcn. renormalization
- Källén-Lehmann spectral density
- Began discussion of renormalization conditions

Today:

- Counterterms, superficial degree of divergence
- Optical theorem
- Refresher on renormalization in QED + Ward identity

Infinities in loop integrals must be regulated.

Can introduce counterterms: cancel off <u>only</u> the UV divergence that arises in loop integrals.

- need more conditions to determine the physical behavior of the QFT after removing these infinities

These are **renormalization conditions**

How many counterms are needed?

Can categorize interacting theories into renormalizeable, super-renormalizable & non-renormalizable.

In general, renormalizable & super-renormalizable theories require finite # of counter-terms $\Rightarrow$ all UV divergences are cancelled by finite # of counterterms $\Rightarrow$ equating (roughly) finite # of counterterms to physical measurements $\Rightarrow$ get a predictive theory.

Thm: Bogoliubov, Parasiuk, Hepp, Zimmermann proved that all UV divergences can be removed by counterterms corresponding to superficial degree of divergence of 1PI amplitudes.

[see Schwartz p. 385, Weinberg, Vol 1, p. 512-3]

Ensures that once we take of 1PI amplitudes, all types of UV divergences are taken care of.

P&S Chap. 10

Evaluate (in QED) superficial degree of divergence.

P&S
Chap. 10.

Evaluate (in QED) superficial degree of divergence.

Highest power of UV scale that could show up in matrix element.

QED, in 4D: N_e external e, N_γ ext. photons,
P_e propagating e, P_γ propagating γ,
V vertices, L loops.

Every matrix element is characterized by this counting.

Superficial deg. of divergence

$$D = 4L - P_e - 2P_\gamma$$

Every loop $\int_0^{\Lambda_{UV}} \frac{d^4k}{(2\pi)^4} \leftarrow$ 4 powers of Λ_{UV}

propagating e: $\frac{1}{(k+p)^2 - m^2} \cdot i(\not k + \not p + m)$: -1 power of Λ_{UV}

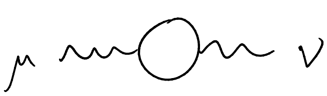
propagating γ: $\frac{1}{(k+p)^2 - 0} \cdot -i g_{\mu\nu}$: -2 powers of Λ_{UV}

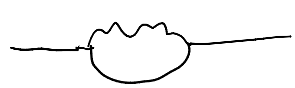
[general ξ: $\frac{1}{(k+p)^2 - 0}(-i)\left(g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}(1-\xi)\right)$ → should $(k+p)$

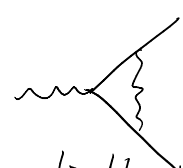
– in Spontaneously broken gauge theories,

$$\frac{1}{(k\ \ )^2 - m^2}(-i)\left(g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2 - \xi m^2}(1-\xi)\right)$$]

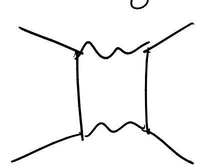
Recall 1-loop QED.







These are all the diagrams for renormalizable operators.



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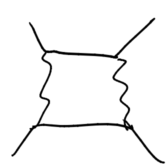
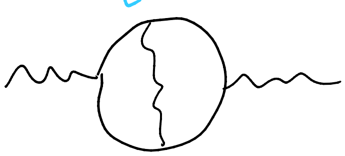


Exercise:

2-loop QED.

Outer loop:
Fermion: or fermion+photon (not only photon)



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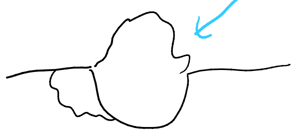
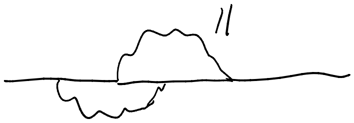
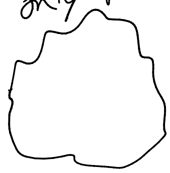




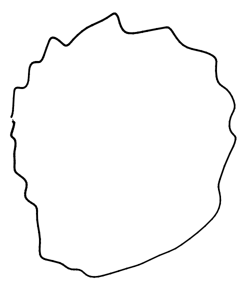
Aside:

only photon:

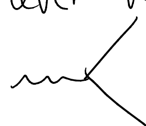


No possible way to attach external legs.

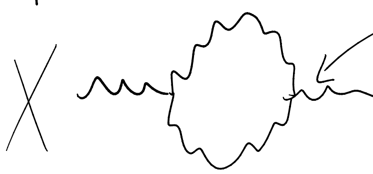
Suppose



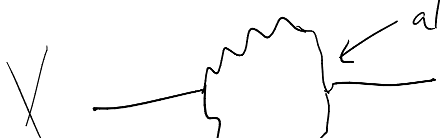
Trying to connect external legs to a photon loop. These have to be tree-level vertices.



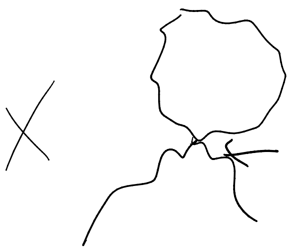
Not possible to do



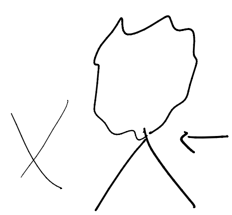
does not exist



also does not exist



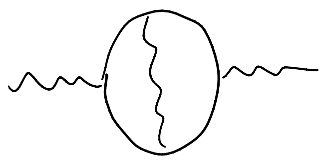
vertex of 4 photons does not exist



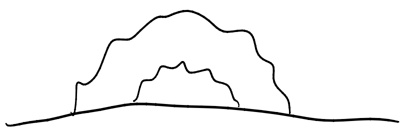
$2\gamma, 2e$ does not exist

During break.

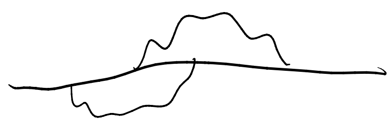
During break,
calculate D for



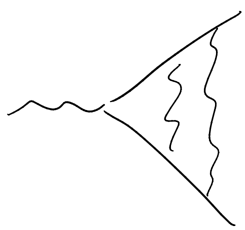
$D=2$. $\displaystyle\int \frac{d^4k_1\, d^4k_2}{(2\pi)^8} \left(\frac{\not{k}}{k^2-m^2}\right)^2 \left(\frac{\not{k}}{k^2-m^2}\right)^2 \cdot \frac{1}{k^2}$



$D=1$ $\displaystyle\int d^4k_1\, d^4k_2 \cdot \frac{1}{k^2}\, \frac{1}{k^2} \cdot \left(\frac{1}{\not{k}}\right)^3$



$D=1$



$D=0$ $\displaystyle\int d^4k_1\, d^4k_2 \cdot \frac{1}{k^2} \cdot \frac{1}{k^2} \cdot \left(\frac{1}{\not{k}}\right)^4$

$$D = 4L - P_e - 2P_\gamma$$

$$L = P_e + P_\gamma - V + 1$$

Each prop. contributes momentum integral
Each vertex $\delta^{(4)}$
One momentum conservation is overall (+ redundant).

$$V = 2P_\gamma + N_\gamma = \tfrac{1}{2}(2P_e + N_e).$$

$$D = 4 - N_\gamma - \tfrac{3}{2} N_e.$$

[for # of dim. d, $D = d + \left(\frac{d-4}{2}\right)V - \left(\frac{d-2}{2}\right)N_\gamma - \left(\frac{d-1}{2}\right)N_e$]

For every N_γ + N_e, only decrease D.

[illegible]

For every N_γ + N_e, only decrease D.
Corresponds with strength of operator given the set of external legs.

Super-renormalizable theories:
only finite # of diagrams superficially diverge.
$\Rightarrow$ Coupling of operator has pos. mass dimn.

renormalizable theories:
only finite # of amps. diverge, but divergences occur at all orders of pert. theory.
$\Rightarrow$ Coupling is dimn-less.

non-renormalizeable theories:
All amps. at sufficiently high order in pert. theory diverge.
$\Rightarrow$ coupling has neg. mass dimn.

Example: $m\bar{\Psi}\Psi$: super-renormalizabe op.

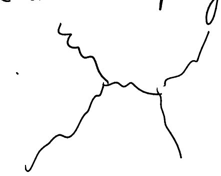
$\bar{\Psi} e \gamma_\mu A^\mu \Psi$: renormalizable

$\frac{1}{\Lambda^2}\bar{\Psi}\gamma^\mu\Psi \; \bar{\Psi}\gamma_\mu\Psi$: non-renormalizable.

?

Gravity: $S = \exp\left[-\int d^4x \sqrt{-g}\left[R^{\alpha\beta}R_{\alpha\beta} + R + R^{\mu\nu\rho\sigma}R_{\mu\nu}R_{\rho\sigma}\right]\right.$

Gravity is not renormalizable: not related to G = Newton's const but instead to spin of graviton (spin 2). Gravitational couplings are dimensionless, but gravitational self-interactions are $\frac{1}{\Lambda^2}$ or $\frac{1}{\Lambda^4}$.



$$S = \exp\left[-\int d^4x \sqrt{-g}\left[R^{\alpha\beta}R_{\alpha\beta} + \bar{\Psi}(i\gamma - m)\Psi\right]\right]$$

metric tensor induces interaction between $G^{\mu\nu}$ graviton + fermion.

Connect to ideas of loop-tree duality.

Topic: Optical Theorem.

Build up loop diagrams (in perturbative expansion) by series of higher order tree-level interactions.

Fundamental concept of QFT: unitarity of S-matrix.

$$S^\dagger S = \mathbb{1}$$

$$S = 1 + iT, \quad T \text{ not Hermitian.}$$

$$\left(\text{Recall} \quad \langle f | T | i \rangle = (2\pi)^4 \delta^{(4)}(p_i - p_f)\, \mathcal{M}(i \to f) \right)$$

$$1 = (1 - iT^\dagger)(1 + iT) \Rightarrow i(T^\dagger - T) = T^\dagger T.$$

$$\langle f | i(T^\dagger - T) | i \rangle = i \langle i | T | f \rangle^* - i \langle f | T | i \rangle$$

$$= i (2\pi)^4 \delta^{(4)}(p_i - p_f) \left(\mathcal{M}^*(f \to i) - \mathcal{M}(i \to f) \right)$$

$$\langle f | T^\dagger T | i \rangle =$$

using completeness relation

$$\sum_X \prod_{j \in X} \int \frac{d^3 p}{(2\pi)^3} \frac{1}{2E_j} \langle f | T^\dagger | X \rangle \langle X | T | i \rangle$$

(X: single & multiparticle states of Hilbert space)

$$= \sum_X (2\pi) \delta^{(4)}(p_f - p_X) (2\pi)^4 \delta^{(4)}(p_i - p_X) \prod_{j \in X} \int \frac{d^3 p_j}{(2\pi)^3} \frac{1}{2E_j} \mathcal{M}(i \to X)\, \mathcal{M}^*(f \to X)$$

Equating:

Generalized optical theorem:

$$\mathcal{M}(i \to f) - \mathcal{M}^*(f \to i)$$

$= i \sum$ [illegible]

$$\mathcal{M}(i\to f) - \mathcal{M}^*(f\to i)$$
$$= i\sum_X \int \prod_{j\in X} \frac{d^3 p_j}{(2\pi)^3}\frac{1}{2E_j}(2\pi)^4\delta^4(p_i - p_X)\,\mathcal{M}(i\to X)\mathcal{M}^*(f\to X)$$

Interpretation

In pert. theory, LHS is tree-level, linear in coupling.

mismatch in coupling order

RHS is quadratic in coupling @ tree-level.

This corresponds then to loop-level on LHS, then LHS is quadratic in coupling.

Two main observations

1. Imaginary part of loop amplitude is determined by tree-level amplitudes.
2. Unitarity requires loops for interacting theories.

Specialize to same particle:

$$2i\,\mathrm{Im}\,\mathcal{M}(A\to A) = i\sum_X \int \prod_{j\in X}\frac{d^3 p_j}{(2\pi)^3}\frac{1}{2E_j}(2\pi)^4\delta^{(4)}(p_A - p_X)\,|\mathcal{M}(A\to X)|^2$$
$$= 2i\,m_A\sum_X \Gamma(A\to X)$$

So
$$\mathrm{Im}\,\mathcal{M}(A\to A) = m_A\sum_X\Gamma(A\to X) = m_A\Gamma_{tot}$$

Imaginary part of amplitude of exact propagator is the mass of particle × total decay width.

Specialize to 2 particles.

$$\mathrm{Im}\,\mathcal{M}(A_{1,2}\to A_{1,2}) = 2E_{CM}|\vec{p}_i|\sum_X \sigma(A_{1,2}\to X)$$

Imaginary part of forward scattering amplitude is proportional to the total cross section.

Optical theorem: imaginary parts of loop amplitudes correspond to intermediate particles going on-shell.

Discontinuity of amplitudes as fcn of ...

correspond to intermediate particles going on-shell.

Discontinuity of amplitudes as fcn. of complex momenta is given by cutting rules (Cutkosky, 1960).

Next time:

Review QED renormalization + Ward identity

Begin $\lambda\phi^4$ renormalization.