QFT II

Lecture 3

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Felix Yu

Last time: More LSZ reduction.

Wavefcn. renormalization.

Källen-Lehmann spectral density as generalization of 1-particle 2-pt. correlation fcn. for interacting theories.

Renormalization conditions.

Today: Categories of renormalizability

Superficial / Naïve degree of divergence

Optical theorem + connection between loops + tree in interacting theories.

In renormalized perturbation theory, we include counterterms with Feynman rules; fix counterterms by renormalization conditions.

How many counterterms are needed?

In general, renormalizeable theories require finite # of counterterms.

As a corollary, all UV divergences can be cancelled with finite # of counterterms.

Schwartz p. 385

Weinberg Vol. 1, p. 512-3

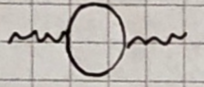
Remark: Bogoliubov, Parasiuk, Hepp, Zimmermann proved all UV divergences can be removed by counterterms corresponding to superficial divergences of 1PI amplitudes.

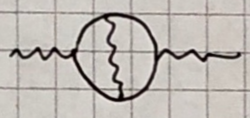
Following Peskin + Schroeder, Chap. 10, we can count UV divergences by external legs + loops. This is the superficial degree of divergence.

QED, 4D: N_e external e, N_γ external photons

P_e prop. e, P_γ prop. photons, V vertices, L loops.

Lecture exercise: Draw 1-loop & 2-loop QED diagrams & sketch the loop momenta integrals.


$$D = 2 \Leftarrow \int d^4k \left(\frac{\not k + m}{k^2 - m^2}\right)^2$$


$$\int d^4k_1 \, d^4k_2 \left(\frac{\not k + m}{k^2 - m^2}\right)^4 \frac{1}{k^2} \Rightarrow D = 2$$

$$D = 4L - P_e - 2P_\gamma$$
= superficial deg. of divergence.
$$L = P_e + P_\gamma - V + 1$$

Each prop. contributes momentum integral. Each V gives $\delta^{(4)}$. One overall momentum conservation (adds 1 back).

$$V = 2P_\gamma + N_\gamma = \tfrac{1}{2}(2P_e + N_e)$$
$$D = 4 - N_\gamma - \tfrac{3}{2}N_e \quad \text{or} \quad D = d + \left(\frac{d-4}{2}\right)V - \left(\frac{d-2}{2}\right)N_\gamma - \left(\frac{d-1}{2}\right)N_e$$
for d dimensions.

Operators and Lagrangians can be categorized by divergence behavior.

1. Super-renormalizeable:
   Only finite # of diagrams superficially diverge
   $\Leftrightarrow$ coupling has pos. mass dimension. $\left(\mathcal{L} = m\bar\Psi\Psi, \ \tfrac{1}{2}m^2\phi^2\right)$
2. Renormalizeable:
   Only finite of amps. diverge, but divergences occur at all orders in pert. theory $\Leftrightarrow$ coupling is dim-less.
3. Non-renormalizeable: All amps. at sufficiently high order in pert. theory. $\Leftrightarrow$ coupling const. has neg. mass dimension.

Optical Theorem:

Another deep connection between tree + loop amplitudes occurs by the unitarity requirement of the S-matrix.

Recall $S = \mathbb{1} + iT$, where $S^\dagger S = \mathbb{1}$. Note T is not Hermitian

$$\mathbb{1} = (1 - iT^\dagger)(1 + iT) = 1 - iT^\dagger + iT + T^\dagger T$$

So $\quad i(T^\dagger - T) = T^\dagger T$

Then, $\langle f | i(T^\dagger - T) | i \rangle = i \langle i | T | f \rangle^* - i \langle f | T | i \rangle$

$$= i (2\pi)^4 \delta^{(4)}(p_i - p_f) \left(\mathcal{M}^*(f \to i) - \mathcal{M}(i \to f) \right)$$

Use completeness relationship

$$\langle f | T^\dagger T | i \rangle = \sum_{\underset{\substack{\uparrow \\ \text{single + multiparticle states of Hilbert space}}}{x}} \prod_{j \in x} \int \frac{d^3 p_j}{(2\pi)^3} \frac{1}{2E_j} \langle f | T^\dagger | x \rangle \langle x | T | i \rangle$$

$$= \sum_x (2\pi)^4 \delta^{(4)}(p_f - p_x) (2\pi)^4 \delta^{(4)}(p_i - p_x) \prod_{j \in x} \int \frac{d^3 p_j}{(2\pi)^3} \frac{1}{2E_j} \mathcal{M}(i \to x) \mathcal{M}^*(f \to x)$$

Equating from above, we get the generalized optical theorem:

$$\mathcal{M}(i \to f) - \mathcal{M}^*(f \to i)$$
$$= i \sum_x \int \prod_{j \in x} \frac{d^3 p_j}{(2\pi)^3} \frac{1}{2E_j} (2\pi)^4 \delta^{(4)}(p_i - p_x) \mathcal{M}(i \to x) \mathcal{M}^*(f \to x)$$

Interpretation:

Consider LHS and RHS in context of perturbation theory,

LHS is linear in coupling @ tree-level, quadratic @ loop-level.

RHS is quadratic @ tree-level.

Equivalence holds order by order in coupling $\Rightarrow$ Imaginary parts of loop amplitudes are determined by tree-level amps.

Bluntly, unitarity <u>requires</u> loops for interacting theories.

We can specialize to same particle.

$$2i \, \text{Im} \, \mathcal{M}(A \to A) = i \sum_X \int \prod_{j \in X} \frac{d^3 p_j}{(2\pi)^3} \frac{1}{2E_j} (2\pi)^4 \delta^{(4)}(p_A - p_X) |\mathcal{M}(A \to X)|^2$$

$$= 2 i m_A \sum_X \Gamma(A \to X)$$

We get $\text{Im} \, \mathcal{M}(A \to A) = m_A \sum_X \Gamma(A \to X) = m_A \Gamma_{tot}$

Interpretation: Imaginary part of amplitude for exact propagator is equal to mass times decay rate

Recall $$- - - - = \frac{i}{p^2 - m_R^2 + \Sigma(p^2) + i\epsilon}$$

Want $m_p^2 - m_R^2 + \Sigma(p^2 = m_p^2) = 0$

$$\frac{1}{m_p} \text{Im} \, \Sigma(p^2 = m_p^2) + \ldots = \Gamma_{tot} = \frac{1}{m_p} \text{Im}(- - \bullet - -)$$

Refine $m_p^2 - m_R^2 + \text{Re} \, \Sigma(p^2 = m_p^2) = 0$

for real pole mass / Breit-Wigner mass.

Near pole, $\Gamma_{tot} \ll m_p$ & $\frac{i}{p^2 - m_p^2 + i m_p \Gamma} \Rightarrow \sigma \sim \frac{g^4}{(p^2 - m_p^2)^2 + (m_p \Gamma)^2}$

Contrast complex pole mass:

$$m_C^2 - m_R^2 + \Sigma(m_C^2) = 0$$

Contrast $\overline{MS}$: set finite parts of counterterms to 0.

We can also specialize to two-particle state.

$$\text{Im} \, \mathcal{M}(A_1 A_2 \to A_1 A_2) = 2 E_{CM} |\vec{p}_i| \sum_X \sigma(A_{1,2} \to X)$$

Imaginary part of <u>forward</u> <u>scattering</u> amplitude is proportional to total cross section. [Forward scattering is same initial & final state.]

Since imaginary parts of loop amplitudes correspond to intermediate particles going on-shell, the optical theorem provides tool for calculating amplitudes. Very active area of research these days in algebraic properties of amps. Example: Discontinuity of amp. as fcn. of complex momenta given by cutting rules / Cutkosky, 1960).