Lecture 2, April 22, 2020

Monday, April 20, 2020 10:49 PM

From last time, main theme was understanding the complications of studying interacting QFTs.  is divergent!

Renormalization is a consequence of regularization of UV divergences.

Introduced LSZ reduction formula:

Begin to tackle the question of how do we relate free eigenstates with interacting eigenstates.

Need to know/derive this overlap between interacting & free eigenstates.

This is called the wavefunction renormalization.

Start with

$$\langle \vec{p}_1 \vec{p}_2 \ldots | iT | \vec{k}_A \vec{k}_B \rangle = (2\pi)^4 \delta^{(4)}(k_A + k_B - \sum p_f)\, i\mathcal{M}(k_A k_B \to p_f)$$

Recall $S = \mathbb{1} + iT$.

We want relationship between in & out states

Relating plane wave states of H & plane wave states of H_0.

Can start with $|0\rangle$, vacuum of free theory.

$$e^{-iHT}|0\rangle = \sum_n e^{-iE_n T}|n\rangle\langle n|0\rangle$$

Assuming $H = H_0 + H_I$, H_I small perturbation & $\langle \Omega | 0 \rangle \neq 0$,

so

$$e^{-iHT}|0\rangle = e^{-iE_0 T}|\Omega\rangle\langle\Omega|0\rangle + \sum_{n\neq 0} e^{-iE_n T}|n\rangle\langle n|0\rangle$$

where $E_0 \equiv \langle\Omega|H|\Omega\rangle$, $E_n > E_0$ for $n \neq 0$, so

take $T \to \infty(1 - i\epsilon)$

$$\Rightarrow |\Omega\rangle = \lim_{T\to\infty(1-i\epsilon)} \left(e^{-iE_0 T}\langle\Omega|0\rangle\right)^{-1} e^{-iHT}|0\rangle$$

Recall LSZ reduction formula,

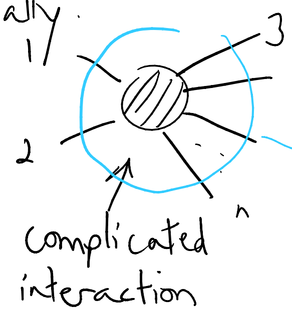
For N-pt. correlation fcns,

$$\langle f|S|i\rangle = \langle p_3 \cdots p_n|S|p_1 p_2\rangle$$

$$= \left[i \int d^4x_1\, e^{-ip_1 x_1} (\Box_1 + m^2) \right] \cdots \left[i \int d^4x_n\, e^{ip_n x_n} (\Box_n + m^2) \right]$$
$$\cdot \langle \Omega | T\{ \phi(x_1) \cdots \phi(x_n) \} | \Omega \rangle$$

Interpretation: When all asymptotic states go on-shell, (well-separated wavepackets with definite momenta), the coefficient of the multiple pole is the S-matrix element.

Pictorially:



All external scalars.
Extract outgoing states + ingoing states.
operate on this n-pt. correlation fcn. by n KG operators.
This gives the amplitude as coeff. of multiple pole.

Key feature: Extracting poles of full n pt. correlation fcn. asymptotic states requires overlap of full H eigenstates w/ free H momentum eigenstates.

[Aside: Every scattering event is strength of some correlation fcn. with given # of particles = amplitude of process.]

This overlap is the wavefcn. renormalization.

(Section 7.1 of P+S) Can motivate the calculation of $\langle p | \phi(0) | 0 \rangle = 1$
vs. $\langle p | \phi(0) | \Omega \rangle = \sqrt{Z}$

ϕ creates momentum eigenstate $|p\rangle$.

Consider $\langle \Omega | T \phi(x) \phi(y) | \Omega \rangle$

Let $|\lambda_0\rangle$ be eigenstates of $\mathbb{P}$ with 0 momentum.

All boosts of $|\lambda_0\rangle$ are eigenstates + have all possible 3-momenta. Can obtain any eigenstate of H with

... are eigenstates & have all possible 3-momenta. Can obtain any eigenstate of H with definite momenta by boosting $|\lambda_0\rangle$.

Completeness

$$(\mathbb{1})_{\text{1-particle}} = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_{\vec{p}}} |\vec{p}\rangle\langle\vec{p}|$$

Let $E_{\vec{p}} \equiv \sqrt{|\vec{p}|^2 + m_\lambda^2}$ with m_λ = energy of $|\lambda_0\rangle$.

Let $|\lambda_{\vec{p}}\rangle$ be boost of $|\lambda_0\rangle$ with $\vec{p}$.

So
$$\mathbb{1} = |\Omega\rangle\langle\Omega| + \underbrace{\sum_\lambda}_{\substack{\text{sum over}\\ |\lambda\rangle \text{ states.}}} \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_{\vec{p}}(\lambda)} |\lambda_{\vec{p}}\rangle\langle\lambda_{\vec{p}}|$$

Insert $\mathbb{1}$ into $\langle\Omega| T\phi(x)\phi(y)|\Omega\rangle$.

$$\langle\Omega|T\phi(x)\left[|\Omega\rangle\langle\Omega| + \sum_\lambda \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_{\vec{p}}(\lambda)} |\lambda_{\vec{p}}\rangle\langle\lambda_{\vec{p}}|\right]\phi(y)|\Omega\rangle$$

$= [$drop the $\langle\Omega|\phi(x)|\Omega\rangle\langle\Omega|\phi(y)|\Omega\rangle]$

Vacuum expectation value of ϕ

$$= \sum_\lambda \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_{\vec{p}}(\lambda)} \langle\Omega|\phi(x)|\lambda_{\vec{p}}\rangle\langle\lambda_{\vec{p}}|\phi(y)|\Omega\rangle$$

Calculate inner product:

$$\langle\Omega|\phi(x)|\lambda_{\vec{p}}\rangle = \langle\Omega| e^{iP\cdot x}\phi(0)e^{-iP\cdot x}|\lambda_{\vec{p}}\rangle$$

P operator

$$= \langle\Omega|\phi(0)|\lambda_{\vec{p}}\rangle e^{-ip\cdot x}\Big|_{p^0=E_{\vec{p}}}$$

Insert $U^{-1}U$, for U being Lorentz boost $\vec{p}$ to 0.

$$= \langle\Omega|U^{-1}\underbrace{U\phi(0)U^{-1}}_{\text{scalar inv. under L.T.}}U|\lambda_{\vec{p}}\rangle e^{-ip\cdot x}\Big|_{p^0=E_{\vec{p}}}$$

$$= \langle\Omega|\phi(0)|\lambda_0\rangle e^{-ip\cdot x}\Big|_{p^0=E_{\vec{p}}}$$

Insert into T.O. product

$$\langle\Omega|T\phi(x)\phi(y)|\Omega\rangle = \sum_\lambda \int \frac{d^4p}{(2\pi)^4} \frac{i}{p^2 - m_\lambda^2 + i\epsilon} e^{-ip\cdot(x-y)} |\langle\Omega|\phi(0)|\lambda_0\rangle|^2$$

$$\langle\Omega|T\phi(x)\phi(y)|\Omega\rangle = \sum_\lambda \int \frac{d^4p}{(2\pi)^4} \frac{i}{p^2-m_\lambda^2+i\epsilon} e^{-ip\cdot(x-y)} |\langle\Omega|\phi(0)|\lambda_0\rangle|^2$$

+ introduce $\int dp^0$ integration to replace $\Big|_{p^0=E_{\vec{p}}}$.

As expected, we get $D(x-y)$ propagator; simple pole at $p^2=m_\lambda^2$.

Break for ~10 min.

First, $|\langle\Omega|\phi(0)|\lambda_0\rangle|^2 = Z$.

From LSZ, this renormalization factor is the residue we extract in 2 pt. fcn.

Z is the field strength renormalization, probability for $\phi(0)$ to create given state from vacuum.

Note also pole location is m_λ, which is physical mass of ϕ.

Second,

$$\langle\Omega|T\phi(x)\phi(y)|\Omega\rangle = \int_0^\infty \frac{dM^2}{(2\pi)} \rho(M^2)\, D(x-y, M^2)$$

M^2 is "probe" of $\rho(M^2)$.

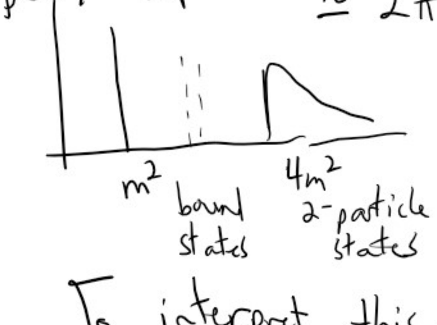
Can think of $D(x-y, M^2)$ as the equivalent of δ-fcn in the convolution integral to decompose ρ.

Källen-Symanzik spectral rep.

$$\rho(M^2) = \sum_\lambda (2\pi)\, \delta(M^2-m_\lambda^2)\, |\langle\Omega|\phi(0)|\lambda_0\rangle|^2$$

$$\simeq 2\pi\, \delta(M^2-m_\lambda^2) Z + \sim M^2 \gtrsim 4m^2 + \text{bound states}$$

$\uparrow$ multiparticle states



To interpret this, let us identify the difference $\langle\Omega|T\phi\phi|\Omega\rangle$ and $\langle 0|T\phi\phi|0\rangle$.

$$\int d^4x\, e^{ip\cdot x} \langle 0|T\phi(x)\phi(y)|0\rangle = \frac{i}{p^2-m^2+i\epsilon}$$

$$\frac{\quad}{p^2 - m^2 + i\epsilon}$$

$$\int d^4x \, e^{ip\cdot x} \langle \Omega | T\phi(x)\phi(y) | \Omega \rangle = \frac{iZ}{p^2 - m_\lambda^2 + i\epsilon} + \int_{\sim 4m^2}^{\infty} \frac{dM^2}{(2\pi)} \rho(M^2) \frac{i}{p^2 - M^2 + i\epsilon}$$

Get to recover the original scalar propagator by introducing renormalized field $\phi_R \equiv Z^{-1/2} \phi_B$.

Also need δm^2 s.t. $m_R^2 = m_B^2 + \delta m^2$.

This gives the origin of wavefcn. renorm.

For fermions,

$$\int d^4x \, e^{ip\cdot x} \langle \Omega | T\Psi(x) \bar{\Psi}(0) | \Omega \rangle = i \frac{Z_2(\not{p} + m)}{p^2 - m^2 + i\epsilon} + \cdots$$

$$\langle \Omega | \Psi(0) | p,s \rangle = \sqrt{Z_2}\, u^s(p)$$

$$\langle \Omega | \bar{\Psi}(0) | p,s \rangle = \sqrt{Z_2}\, \bar{v}^s(p)$$

Difference b/t renormalized field + bare field + role of WFR:

Weinberg, p. 438-9

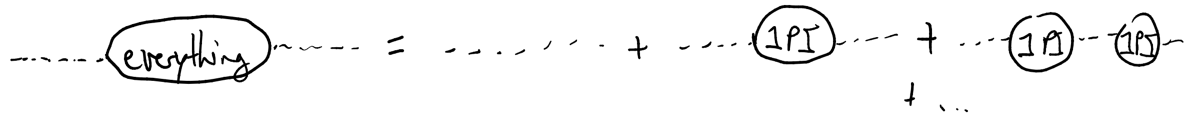
A renormalized field is one whose propagator has the same behavior near its pole as for a free field, and the renormalized mass is defined by the position of the pole.

Preview: These serve as renormalization conditions.

Quick recap of 1PI:

Note $\langle \Omega | T\phi(x)\phi(y) | \Omega \rangle$ fixes scalar fields at x + y.

Motivates



Consider $\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_1$

true renormalized ϕ

$$\mathcal{L}_0 = \frac{1}{2}(\partial_\mu \phi)^2 - \frac{1}{2} m^2 \phi^2$$

$$\mathcal{L}_0 = \frac{1}{2}(\partial_\mu \phi)^2 - \frac{1}{2} m^2 \phi^2$$

$$\mathcal{L}_1 = \frac{1}{2}(Z-1)\left((\partial_\mu\phi)^2 - m^2\phi^2\right) + \frac{1}{2} Z \delta m^2 \phi^2 - V(\phi)$$

$$\phi_R \equiv \frac{\phi_B}{\sqrt{Z}}, \quad m^2 = m_0^2 + \delta m^2, \quad V(\phi) \equiv V_B(\sqrt{Z}\,\phi)$$

Recall $-i\Pi^*(p^2) = $ --- (1PI) --- omitting external propagator factors

Gives --- (everything) --- $= \dfrac{i}{p^2-m^2} + \dfrac{i}{p^2-m^2}\left(-i\Pi^*\right)\dfrac{i}{p^2-m^2} + \ldots$

$$= \frac{i}{p^2 - m^2 - \Pi^*(p^2)}$$

Can get $\Pi^*(p^2) = (Z-1)(p^2-m^2) + Z\,\delta(m^2) + \Pi^*_{loop}(p^2)$

Generates 2 renormalization conditions:

① Require $\Pi^*(p^2=m^2) = 0$ so pole is $p^2 = m^2$.

② Need unit residue $\dfrac{d}{dp^2}\Pi^*(p^2)\Big|_{p^2=m^2} = 0$.

Solve $Z\,\delta m^2 = -\Pi^*_{loop}(m^2)$

$$Z = 1 + \left[\frac{d}{dp^2}\Pi^*_{loop}(p^2)\right]\Big|_{p^2=m^2}$$

Simple interpretation: subtract first-order polynomial in p^2 from $\Pi^*_{loop}(p^2)$ s.t. difference satisfies two conditions. This will incidentally cancel UV divergences in loop integrals!

QED Renormalization proceeds similarly.
(Peskin 7.4 - 7.5)

Remark: Divergences in loop-integrals must be regularized

[illegible]

Remark: Divergences in loop-integrals must be <u>regularized</u> (dim. reg., Pauli-Villars) but the renormalization conditions will introduce counterterms that give finite physical answers <u>independent</u> of regulator.

These counterterms & role in renormalized perturbation theory will be focus of next lecture.