QFT II.

Lecture 2.

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Last time: Emphasized the complications of studying interacting QFTs.
Presented LSZ reduction formula.

Today: Physics / Interpretation of LSZ reduction formula
Motivate wavefcn. renormalization factor
Källén - Lehmann spectral density
Introduction to renormalization conditions

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Recall S-matrix, $S = \mathbb{1} + iT$

We connected S-matrix to matrix element:

$$\langle \vec{p}_1 \vec{p}_2 \ldots | iT | \vec{k}_A \vec{k}_B \rangle = (2\pi)^4 \delta^{(4)}(k_A + k_B - \Sigma p_f)\, i\mathcal{M}(k_A, k_B \to p_f)$$

We then wanted a relation for in + out states
relating plane wave states (eigenstates of $H$) with
plane wave states (eigenstates of $H_0$).

Consider the free vacuum $|0\rangle$ + evolve:

$$e^{-iHT}|0\rangle = \sum_n e^{-iE_n T}|n\rangle\langle n|0\rangle$$

Assume $H = H_0 + H_I$ + $H_I$ a small perturbation such that
$\langle \Omega | 0 \rangle \neq 0$, so

$$e^{-iHT}|0\rangle = e^{-iE_0 T}|\Omega\rangle\langle\Omega|0\rangle + \sum_{n\neq 0} e^{-iE_n T}|n\rangle\langle n|0\rangle$$

$$E_0 \equiv \langle \Omega | H | \Omega \rangle$$

Since $E_n > E_0$ for $n \neq 0$, we can take $T \to \infty(1-i\epsilon)$ + solve for $|\Omega\rangle$

$$\Rightarrow |\Omega\rangle = \lim_{T\to\infty(1-i\epsilon)} \left(e^{-iE_0 T}\langle\Omega|0\rangle\right)^{-1} e^{-iHT}|0\rangle$$

Recall LSZ for n-pt. correlation fcns.

$$\langle f|S|i\rangle = \langle p_3 \ldots p_n | S | p_1 p_2\rangle$$
$$= \left[ i\int d^4x_1\, e^{-ip_1 \cdot x_1} (\Box_1 + m^2)\right] \ldots \left[ i\int d^4 x_n\, e^{ip_n \cdot x_n} (\Box_n + m^2)\right]$$
$$\langle \Omega | T\{\phi(x_1)\ldots\phi(x_n)\} | \Omega\rangle$$

When all asymptotic states (i.e. well separated wavepackets with def. momenta) go on-shell, the coefficient of the multiple pole is the S-matrix element.

Key feature: Extracting poles of full n pt. correlation fcn. that correspond to asymptotic states requires overlap of full Hamiltonian momentum eigenstates with free H eigenstates.

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Motivating wavefcn renormalization

Peskin + Schroeder 7.1.

We now motivate the calculation of $\langle p|\phi(0)|\Omega\rangle \equiv \sqrt{Z}$, compared to $\langle p|\phi(0)|0\rangle = 1$.

Consider: $\langle \Omega | T\phi(x)\phi(y)|\Omega\rangle$

Let $|\lambda\rangle$ be eigenstate of $\mathbb{P}$ with 0 momentum.

All boosts of $|\lambda_0\rangle$ are eigenstates + have all possible 3-momenta.

$\Rightarrow$ Can obtain any eigenstates of H with definite momentum by boosting $|\lambda_0\rangle$.

Completeness relation

$$(\mathbb{1}_{1\text{-particle}}) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{(2E_{\vec{p}})} |\vec{p}\rangle\langle\vec{p}|$$

Let $E_{\vec{p}} \equiv \sqrt{|\vec{p}|^2 + m_\lambda^2}$ with $m_\lambda = E$ of $|\lambda_0\rangle$

+ let $|\lambda_{\vec{p}}\rangle$ be boost of $|\lambda_0\rangle$ with $\vec{p}$.

So $\mathbb{1} = |\Omega\rangle\langle\Omega| + \sum_\lambda \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_{\vec p}(\lambda)} |\lambda_{\vec p}\rangle\langle\lambda_{\vec p}|$

(sum over $|\lambda_0\rangle$ states)

Insert $\mathbb{1}$ into $\langle\Omega| T\phi(x)\phi(y)|\Omega\rangle$.

Will drop $\langle\Omega|\phi(x)|\Omega\rangle \langle\Omega|\phi(y)|\Omega\rangle$ (vacuum exp. value)

For $x^0 > y^0$, $\langle\Omega|\phi(x)\phi(y)|\Omega\rangle$

$$= \sum_\lambda \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_{\vec p}(\lambda)} \langle\Omega|\phi(x)|\lambda_{\vec p}\rangle\langle\lambda_{\vec p}|\phi(y)|\Omega\rangle$$

$$\langle\Omega|\phi(x)|\lambda_{\vec p}\rangle = \langle\Omega| e^{iP\cdot x}\phi(0)e^{-iP\cdot x}|\lambda_{\vec p}\rangle$$

$P$ operator

$$= \langle\Omega|\phi(0)|\lambda_{\vec p}\rangle e^{-ip\cdot x}\big|_{p^0=E_{\vec p}}$$

$U$ is Lorentz boost $\vec p$ to $0$

$$= \langle\Omega| U^{-1}U\phi(0)U^{-1}U|\lambda_{\vec p}\rangle e^{-ip\cdot x}\big|_{p^0=E_{\vec p}}$$

$$= \langle\Omega|\phi(0)|\lambda_0\rangle e^{-ip\cdot x}\big|_{p^0=E_{\vec p}}$$

$\phi$ is scalar, inv. under Lorentz trans.

$|\Omega\rangle$ is vacuum, unaffected by Lorentz boost

[cf. P&S, eqn. 2.54] Introduce $p^0$ integration

$$\langle\Omega|\phi(x)\phi(y)|\Omega\rangle = \int_0^\infty \frac{dM^2}{(2\pi)} \rho(M^2)\, D(x-y; M^2)$$

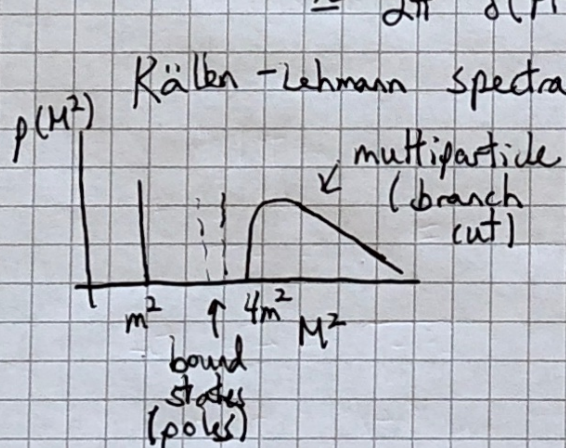
$M^2$ is "probe" of $\rho(M^2)$

$$\rho(M^2) = \sum_\lambda (2\pi)\,\delta(M^2 - m_\lambda^2)\, |\langle\Omega|\phi(0)|\lambda_0\rangle|^2$$

$$\simeq 2\pi\, \delta(M^2 - m_\lambda^2)\, Z + \text{bound states} + \sim M^2 \gtrsim 4m^2$$

multiparticle states

Källen-Lehmann spectral rep.




To recover original scalar propagator, introduce renormalized field $\phi_R \equiv Z^{-1/2}\phi_B$ + $\delta m^2$ s.t. $m_R^2 = m_B^2 + \delta m^2$.

Citing Weinberg : p. 438-9 [QFT Vol. I]

A renormalized field is one whose propagator has the same behavior near its pole as for a free field, and the renormalized mass is defined by the position of the pole.
⇒ Serve as renormalization conditions.

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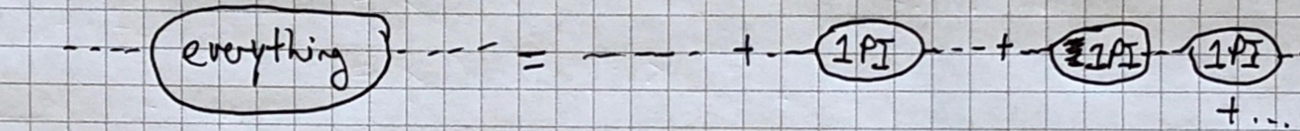
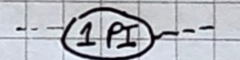
For WFR, recall the decomposition of 1PI (one particle irreducible):

$\langle\Omega| T\phi(x)\phi(y)|\Omega\rangle$ fixes scalar fields at $x$ + $y$.

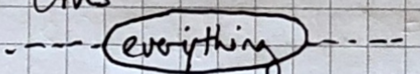
Motivates




Each insertion of 1PI can be calculated perturbatively.

We identify $-i\Pi^*(p^2)$ =  , omitting external propagator factors.

Gives




$$= \frac{i}{p^2-m^2} + \frac{i}{p^2-m}(-i\Pi^*)\frac{i}{p^2-m^2} + \dots$$

$$= \frac{i}{p^2-m^2-\Pi^*}$$

Renormalization conditions:

① Require $\Pi^*(p^2=m^2)=0$, so pole is $p^2=m^2$.

② Want unit residue, so $\frac{d}{dp^2}\Pi^*(p^2)\Big|_{p^2=m^2}=0$.

Consider $\mathcal{L}=\mathcal{L}_0+\mathcal{L}_1$,

free ↙ ↙ renormalized $\phi$

$$\mathcal{L}_0 = \frac{1}{2}(\partial_\mu\phi)^2 - \frac{1}{2}m^2\phi^2$$

$$\mathcal{L}_1 = \frac{1}{2}(Z-1)\left((\partial_\mu\phi)^2 - m^2\phi^2\right) + \frac{1}{2}Z\,\delta m^2\phi^2 - V(\phi)$$

$$V(\phi) \equiv V_0(\sqrt{Z}\,\phi)$$

$\phi = \frac{\phi_0}{\sqrt{Z}}$

$m^2 = m_0^2 + \delta m^2$

Evaluate $\Pi^*(p^2) = (Z-1)(p^2-m^2) + Z\delta m^2 + \Pi^*_{loop}(p^2)$ ⑤

Solve $Z\,\delta m^2 = -\Pi^*_{loop}(m^2)$

$$Z = 1 + \left[\frac{d}{dp^2}\Pi^*_{loop}(p^2)\right]\Big|_{p^2=m^2}$$

Remark: The presence of (unspecified) loops in determining $Z\,\delta m^2$ and $Z-1$ justifies the interaction treatment.

Simple interpretation: Subtract a first-order polynomial in $p^2$ from $\Pi^*_{loop}(p^2)$ s.t. difference satisfies two conditions. This will incidentally cancel UV divergences in loop integrals, but the renormalization of masses and fields has nothing to do with the presence of infinities, and would be necessary even in theories with all momentum loop integrals converged.

Weinberg p. 441.

Trust QED renormalization proceeds similarly (as seen in QFT I. See sections 7.4 + 7.5 of P+S).

Remark: Divergences in loop integrals must be <u>regularized</u> (dim. reg. or Pauli-Villars) but the renormalization conditions will introduce counterterms that give <u>finite</u> physical answers independent of regulator.

This differentiates renormalized perturbation theory w/ Feynman rules involving counterterms from bare pert. theory.