Lecture 1, April 20, 2020

Monday, April 20, 2020 8:13 AM

Welcome to QFT II:

08.128.809. Theoretische Elementarteilchenphysik

Today's plan:

- Course announcements
- Recap of QFT I
- Begin (Continue) renormalization

Physics: all about making quantifiable predictions for physical phenomena.

Classical 
throwing a ball.

$\downarrow$

Quantum: instead: lost possibility to measure every possible property of a particle simultaneously

$[\hat{x}, \hat{p}] = i\hbar \neq 0$

$\downarrow$

Quantum field theory:

In QFT 1: pushing predictive power of quantum systems further: extrapolate to as high E as possible, going to as small distances as possible.

Equations to describe phenomena are based on S-matrix.

Ingredients: unitarity, locality, causality.

Theory: write down field content + matter + symmetries that describe the physical phenomena you want to describe

- prescription: write EOM
  - Solve with particle interactions
    - $\rightarrow$ encode as Feynman diagrams

Use matrix elements from Feynman diagrams to calculate cross sections + decays $\Rightarrow$ compare with experiment

refine interactions + field content

First main unit:

When we try to calculate in perturbative field theory, move beyond tree-level Feynman diagrams + consider loops.

QFT 1 example:

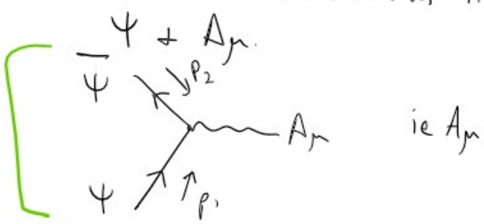
Quantum Electrodynamics:

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu}F^{\mu\nu} + \bar{\Psi}\left(i\not{D} - m\right)\Psi$$

$$\not{D} = \not{\partial} - ie\not{A}, \quad \gamma^\mu \partial_\mu \equiv \not{\partial}$$

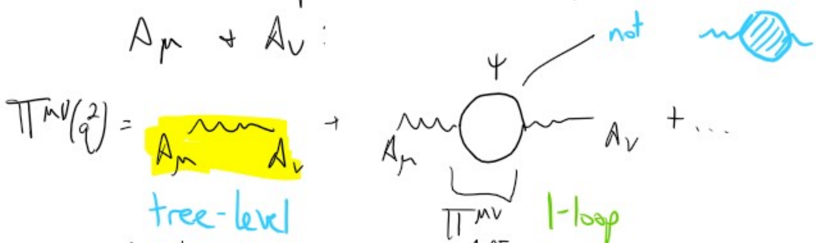
For $e=0$ (gauge coupling), no interactions.
Theory decouples into free vector + fermion.
Solutions are just plane waves.

For $e \neq 0$ $\left(\alpha = \frac{e^2}{4\pi} \approx \frac{1}{137} \text{ @ } Q^2 = 0 \text{ GeV}\right)$
there is a non-trivial interaction between
Ψ + A_μ.



Two comments:

① This nontrivial interaction gives rise to the entire formalism/prescription of renormalization.
For example, consider 2-pt. correlation fcn. between A_μ + A_ν:



Analyzing the loop structure $\Rightarrow \propto \int \frac{d^4k}{(2\pi)^4}\left[\frac{1}{\not{p} - m)}\right]^2$
gives a divergent result.
↳ näive dimensional analysis → quadratically divergent as $k \to \infty$. (UV divergence)

Lost predictivity!

Generic problem in QFT.

(All) Most interactions you add in QFT lead to loop corrections that give divergences.
Main unit of QFT 2: Renormalization solves these divergences and reformulates the theory from "bare" Lagrangian to "renormalized" Lagrangian.

② Nontrivial interaction destroys our ability to

② Nontrivial interaction destroys our ability to solve exactly the QFT.

Nowadays, we have lots of experience & are comfortable with perturbative QFT.

Emphasizes that perhaps there is a different approach, not based on perturbation theory, where we can retain solvability of the quantum system.

Break for $\mathcal{O}(5\text{ min})$

Interacting Theories.

Lehmann-Symanzik-Zimmerman reduction formula

& Källén-Lehmann spectral representation

Starting point for renormalization:

How do you relate the eigenstates of an interacting Hamiltonian vs. the eigenstates of free Hamiltonian?

Notation: $|0\rangle$ is ground state of free theory

$|\Omega\rangle$ is " " of interacting theory.

Previously, object of interest was correlation fcn.

$\langle 0| T\,\phi(x)\,\phi(y)\,|0\rangle \leftarrow$ 2 pt. correlation fcn. in free theory.

Generalize to n- particles

Schwartz, p.70-74.

$\langle \Omega |\, T\phi_1(x_1) \ldots \phi_n(x_n)\,|\Omega\rangle$

$\equiv \langle f\,|\,\mathcal{M}\,|\,i\rangle$ – S-matrix element

Amplitude: $\langle f\,|\,S - \mathbb{1}\,|\,i\rangle = i(2\pi)^4\delta^4(\Sigma p)\,\mathcal{M}$.

final state of n-2 particles, $t \to +\infty$

initial state (2 particles for concreteness) $t \to -\infty$

$|i\rangle, |f\rangle$ asymptotically free states.

$|i\rangle = \sqrt{2E_1}\,\sqrt{2E_2}\;a^\dagger_{p_1}(-\infty)\;a^\dagger_{p_2}(-\infty)\,|\Omega\rangle$

$|f\rangle = \sqrt{2E_3}\,\ldots\sqrt{2E_n}\;a^\dagger_{p_3}(+\infty)\ldots a^\dagger_{p_n}(+\infty)\,|\Omega\rangle$

Assume $|f\rangle \neq |i\rangle$, so discard $\mathbb{1}$.

$\langle f|S|i\rangle = 2^{n/2}\sqrt{E_1 \ldots E_n}\;\langle\Omega|\,a_{p_3}(\infty)\ldots a_{p_n}(\infty)\;a^\dagger_{p_1}(-\infty)\,a^\dagger_{p_2}(-\infty)\,|\Omega\rangle$

$\langle f|S|i\rangle = 2^{n/2}\sqrt{E_1 \cdots E_n}\ \langle\Omega|\, a_{p_3}(\infty) \cdots a_{p_n}(\infty)\ a^\dagger_{p_1}(-\infty)\, a^\dagger_{p_2}(-\infty)|\Omega\rangle$

Need to show:

$$i\int d^4x\ e^{ip\cdot x}(\Box + m^2)\phi(x) = \sqrt{2E_p}\ (a_p(\infty) - a_p(-\infty))$$

for asymptotic free states. Assume fields die off at $\vec{x} = \pm\infty$, so we can use IBP.

$$i\int d^4x\ e^{ip\cdot x}(\Box + m^2)\phi(x) = i\int d^4x\ e^{ip\cdot x}(\partial_t^2 - \vec{\partial}_x^{\,2} + m^2)\phi(x)$$

IBP, require $(\partial_x \phi\, e^{-ip\cdot x})\Big|_{-\infty}^{\infty} = 0$

$$= i\int d^4x\ e^{ip\cdot x}(\partial_t^2 + \vec{p}^{\,2} + m^2)\phi(x)$$

$$= i\int d^4x\ e^{ip\cdot x}(\partial_t^2 + E_p^2)\phi(x)$$

Now, use

$$\partial_t[e^{ip\cdot x}(i\partial_t + E_p)\phi(x)] = (\text{chain rule}) = i e^{ip\cdot x}(\partial_t^2 + E_p^2)\phi(x)$$

independent of B.C.s

$$\Rightarrow i\int d^4x\, e^{ip\cdot x}(\Box + m^2)\phi(x) = \int d^4x\ \partial_t(e^{ip\cdot x}(i\partial_t + E_p)\phi(x))$$

$$= \int dt\ \underbrace{\partial_t\left[e^{iE_p t}\int d^3x\ e^{-i\vec{p}\cdot\vec{x}}(i\partial_t + E_p)\phi(x)\right]}_{\text{Integrand is total derivative, evaluate on bdry } t = \pm\infty}$$

Note $a_p(t)$ & $a_p^\dagger(t)$ operators are time-independent at $t \to \pm\infty$.

Continue calculating:

$$\star \int d^3x\ e^{-i\vec{p}\cdot\vec{x}}(i\partial_t + E_p)\phi(x)$$

$$= \int d^3x\ e^{-i\vec{p}\cdot\vec{x}}(i\partial_t + E_p)\int\frac{d^3k}{(2\pi)^3}\cdot\frac{1}{\sqrt{2E_k}}\left(a_k(t)e^{-ik\cdot x} + a_k^\dagger(t)e^{+ik\cdot x}\right)$$

Pop quiz: $a_k \to$ annihilate momentum mode k

$a_k^\dagger \to$ creates momentum mode k.

$$= \int\frac{d^3k}{(2\pi)^3}\int d^3x\left[\frac{E_k + E_p}{\sqrt{2E_k}}\ a_k(t)e^{-ik\cdot x}e^{-i\vec{p}\cdot\vec{x}} + \frac{(-E_k + E_p)}{\sqrt{2E_k}}\ a_k^\dagger(t)e^{ik\cdot x}e^{-i\vec{p}\cdot\vec{x}}\right]$$

Used $\partial_t a_k(t) = 0$ for bdrys.

Now, $\int d^3x\ e^{-ik\cdot x}e^{-i\vec{p}\cdot\vec{x}} = e^{-iE_k t}\cdot\delta^{(3)}(\vec{p} - \vec{k})(2\pi)^3$ & similar for second term.

$$\star = \sqrt{2E_p}\ a_p(t)\ e^{-iE_p t}$$

So $i\int d^4x\ e^{ip\cdot x}(\Box + m^2)\phi(x) = \int dt\ \partial_t[\ldots]$

$$= \sqrt{2E_p}\left(a_p(\infty) - a_p(-\infty)\right)$$

by h.c.,

$$\sqrt{2E_p}\left(a_p^\dagger(\infty) - a_p^\dagger(-\infty)\right) = -i\int d^4x\, e^{-ip\cdot x}(\Box + m^2)\phi(x)$$

Returning to amplitude:

$$\langle f|S|i\rangle = \sqrt{2^n E_1 \cdots E_n}\,\langle\Omega|\, a_{p_3}(\infty)\cdots a_{p_n}(\infty)\, a_{p_1}^\dagger(-\infty)\, a_{p_2}^\dagger(-\infty)|\Omega\rangle$$

$$= \sqrt{2^n E_1 \cdots E_n}\,\langle\Omega|\, T\{a_{p_3}\cdots a_{p_2}^\dagger\}|\Omega\rangle$$

$$= \sqrt{2^n E_1 \cdots E_n}\,\langle\Omega|\, T\Big\{[a_{p_3}(\infty) - a_{p_3}(-\infty)]\cdots[a_{p_n}(\infty) - a_{p_n}(-\infty)]$$
$$[a_{p_1}^\dagger(-\infty) - a_{p_1}^\dagger(\infty)]\,[a_{p_2}^\dagger(-\infty) - a_{p_2}^\dagger(+\infty)]\Big\}|\Omega\rangle$$

All new $a, a^\dagger$ will shuffle to annihilate on vacuum.

$$\langle f|S|i\rangle = \langle p_3\cdots p_n|S|p_1 p_2\rangle$$

$$= \left[i\int d^4x_1\, e^{-ip_1\cdot x_1}(\Box_1 + m^2)\right]\cdots\left[i\int d^4x_n\, e^{ip_n\cdot x_n}(\Box_n + m^2)\right]$$
$$\langle\Omega|\, T\{\phi(x_1)\cdots\phi(x_n)\}|\Omega\rangle$$

Lehmann-Symanzik-Zimmermann reduction formula

Discussion / Comments:

For given S-matrix, multiply correlation fcn. by $\Box + m^2$ and Fourier transform.

For $\phi(x)$ as free field, $(\Box + m^2)\phi = 0$.

But one-particle states have propagators $\frac{1}{\Box + m^2} = \frac{1}{p^2 - m^2}$

the $\Box + m^2$ operator instead extracts the residue of the single particle pole.

Remark: unitarity requires simple poles, so correlation fcns. can only fall off as p^{-2} for large p^2.

Simple pole: analytic in complex analysis.

Consider amplitude as fcn. of momentum, can only isolated poles or branch cuts + no irreducible poles (counterexample: $e^{1/z}$)

No statement / requirement of perturbativity.

Generalization of two-pt. correlation fcns.

$$\langle \Omega | T\phi(x)\phi(y) | \Omega \rangle$$

$$= \int_0^\infty \frac{dM^2}{2\pi} \rho(M^2)\, D(x-y, M^2)$$

M^2 ($\equiv Q^2$) is the momentum used to probe the correlation.

D is Greens fcn. for KG eqn.

$\rho(M^2)$ is the Källén - Lehmann sprectral density.

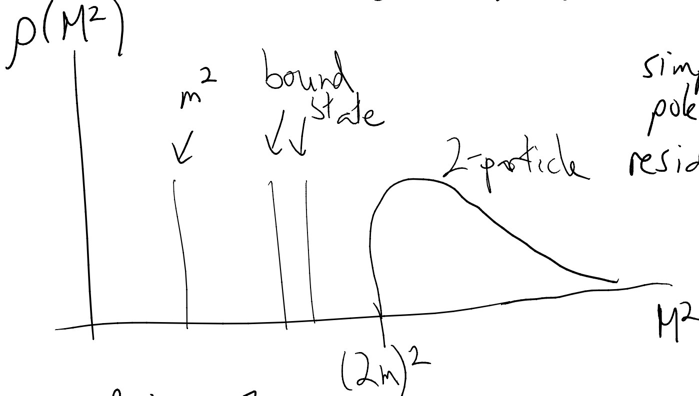
$$\rho(M^2) = \sum_\lambda (2\pi)\, \delta(M^2 - m_\lambda^2)\, |\langle \Omega | \phi(0) | \lambda_0 \rangle|^2$$

In interacting theory, $\rho(M^2) = \delta(M^2 - m^2)\,(2\pi) Z$

simple pole at m^2, residue of Z

$+$ 2-particle states above $(2m)^2$ — branch cut

$+$ bound states — make simple pole.



Peskin fig.7.2.