Lecture 14, June 22, 2020

Monday, June 22, 2020 9:02 AM

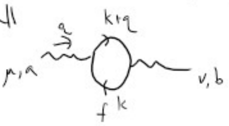
Last time: Focus on calculations in non-Abelian theory that demonstrate necessity of ghosts.

$f\bar{f} \to G^a_\mu G^b_\nu$; began $\Pi_{\mu\nu}(q^2)$ for $G^a G^b$

Office hours: Friday, 9am

Today: Finish $\Pi_{\mu\nu}(q^2)$

Recall



$$i\mathcal{M}_f = \sum_f \frac{i g^2}{16\pi^2}\left(q^2 g^{\mu\nu} - q^\mu q^\nu\right)\delta^{ab}\left(\frac{-8}{3\epsilon}C(r_f)\right)$$

where $C(\square) = C(\text{"fundamental"}) = \frac{1}{2}$.

Now: Gauge diagrams, ghost diagram (+ scalar diagram)

Gauge loops:

$\xi = 1$

$$i\mathcal{M}_3 = \left(\frac{1}{2}\right) g^2 \int\frac{d^4k}{(2\pi)^4}\,\frac{-i}{k^2}\,\frac{-i}{(k+q)^2}\,\left(f^{acd}\right)\Big(g^{\mu\rho}(q-k)^\sigma + g^{\rho\sigma}(2k+q)^\mu + g^{\sigma\mu}(-k-2q)^\rho\Big)$$

symmetry factor

ignore extra Lorentz indices on propagators

$$\times f^{bcd}\left(\delta^\nu{}_\rho(k-q)_\sigma + g_{\rho\sigma}(-2k-q)_\nu + \delta^\nu{}_\sigma(k+2q)_\rho\right)$$

$$f^{acd} f^{bcd} = C_2(G)\,\delta^{ab} = N\,\delta^{ab}$$

$$\frac{1}{k^2(k+q)^2} = \int dx\,\frac{1}{(k^2 + 2xk\cdot q + xq^2)^2}$$

$\ell = k + xq$

$$= \int dx\,\frac{1}{(\ell^2 + xq^2 - x^2q^2)^2}$$

$$\Delta \equiv x^2q^2 - xq^2$$

$$i\mathcal{M}_3 = \frac{-1}{2} g^2 C_2(G)\,\delta^{ab}\int\frac{d^4\ell}{(2\pi)^4}\,\frac{1}{(\ell^2-\Delta)^2}$$

$$\Big[-g^{\mu\nu}(q-k)^2 + (2k+q)^\mu(k-q)^\nu + (k-q)^\mu(-k-2q)^\nu$$
$$+ (q-k)^\mu(-2k-q)^\nu + (2k+q)^\mu(-2k-q)^\nu + (-k-2q)^\mu(-2k-q)^\nu$$
$$+ (k+2q)^\mu(q-k)^\nu + (2k+q)^\mu(k+2q)^\nu + g^{\mu\nu}(-k-2q)\cdot(k+2q)\Big]$$

$$\text{Numerator} = -g^{\mu\nu}\left(5q^2 + 2q\cdot k + 2k^2\right)$$
$$- d\,(q+\underline{2k})^\mu(q+\underline{2k})^\nu$$
$$+ 6\,\underline{k^\mu k^\nu} + 3k^\mu q^\nu + 3q^\mu k^\nu + 6q^\mu q^\nu$$

Shift numerator: $\ell = k + xq$

Drop terms linear in ℓ, $\ell^\mu\ell^\nu = \frac{1}{d}\ell^2 g^{\mu\nu}$

$$= -g^{\mu\nu}\left(2\ell^2 + 4\ell^2 - \frac{6}{d}\ell^2\right)$$
$$- g^{\mu\nu}\left(5q^2 - 2xq^2 + 2x^2q^2\right)$$
$$- d\,\left(q^\mu q^\nu - 4xq^\mu q^\nu + 4x^2q^\mu q^\nu\right)$$

$$-g^{\mu\nu}(5q^2 - 2xq^2 + 2x^2q^2)$$
$$-d\,(q^\mu q^\nu - 4xq^\mu q^\nu + 4x^2 q^\mu q^\nu)$$
$$+ 6q^\mu q^\nu - 6xq^\mu q^\nu + 6x^2 q^\mu q^\nu$$
$$= -g^{\mu\nu}\, 6\left(1-\frac{1}{d}\right)\ell^2$$
$$-g^{\mu\nu} q^2 (5-2x+2x^2)$$
$$-d\, q^\mu q^\nu (1-4x+4x^2) + 6q^\mu q^\nu (1-x+x^2)$$

$$\int \frac{d^d\ell}{(2\pi)^d} \frac{1}{(\ell^2-\Delta)^n} = \frac{(-1)^n i}{(4\pi)^{d/2}} \frac{\Gamma(n-\frac{d}{2})}{\Gamma(n)} \left(\frac{1}{\Delta}\right)^{n-d/2}$$

$$\int \frac{d^d\ell}{(2\pi)^d} \frac{\ell^2}{(\ell^2-\Delta)^n} = \frac{(-1)^{n-1} i}{(4\pi)^{d/2}} \frac{d}{2} \frac{\Gamma(n-\frac{d}{2}-1)}{\Gamma(n)} \left(\frac{1}{\Delta}\right)^{n-\frac{d}{2}-1}$$

$$i\mathcal{M}_3 = \frac{-g^2}{2} C_2(G)\, \delta^{ab}$$
$$\int dx \left[-6\left(1-\frac{1}{d}\right) g^{\mu\nu} \left(\frac{-i}{(4\pi)^{d/2}} \frac{d}{2} \frac{\Gamma(1-\frac{d}{2})}{(\Delta)^{1-d/2}}\right)\right.$$
$$+ \frac{i}{(4\pi)^{d/2}} \frac{\Gamma(2-\frac{d}{2})}{(\Delta)^{2-d/2}} \left(-g^{\mu\nu} q^2 (5-2x+2x^2) \right.$$
$$+ q^\mu q^\nu (1-4x+4x^2)(-d)$$
$$\left.\left. + 6q^\mu q^\nu (1-x+x^2) \right) \right]$$

Note first line has $\Gamma(1-\frac{d}{2}) \cdot \left(\frac{d}{2} - \frac{1}{2}\right)$
which does not change pole structure.

This is a sign that the calculation is incomplete.
In particular, need $\{\}$

$$\Delta = x^2 q^2 - xq^2$$
$$i\mathcal{M}_3 = \frac{-1}{2} g^2 C_2(G)\, \delta^{ab} \int dx \frac{i}{(4\pi)^{d/2}} \left(\frac{1}{\Delta}\right)^{2-d/2}$$
$$\left[6 \left(\frac{d}{2} - \frac{1}{2}\right) g^{\mu\nu} \Gamma(1-d/2)\, q^2 x(x-1) \right.$$
$$+ \Gamma(2-d/2)\, g^{\mu\nu} q^2 (-5+2x-2x^2)$$
$$\left. + q^\mu q^\nu \Gamma(2-d/2) \left(d(-1+4x-4x^2) + 6(1-x+x^2) \right) \right]$$

$\Gamma \propto \xi^k$

$$iM_4 = \frac{1}{2}\int \frac{d^4k}{(2\pi)^4} \frac{-ig_{\rho\sigma}}{k^2} \delta^{cd} (-ig^2)$$
$$\Big[f^{abe} f^{cde} (g^{\mu\rho} g^{\nu\sigma} - g^{\mu\sigma} g^{\nu\rho}) + f^{ace} f^{bde} (g^{\mu\nu} g^{\rho\sigma} - g^{\mu\sigma} g^{\nu\rho}) + f^{ade} f^{bce} (g^{\mu\nu} g^{\rho\sigma} - g^{\mu\rho} g^{\nu\sigma}) \Big]$$
$$= -g^2 C_2(G)\, \delta^{ab} \underbrace{\int \frac{d^4k}{(2\pi)^4} \frac{1}{k^2}}\, g^{\mu\nu}(d-1)$$

To adopt same denominator + shift.

$$= -g^2 C_2(G)\, \delta^{ab} \underbrace{\int \frac{d^4k}{(2\pi)^4} \frac{(q+k)^2}{k^2 (q+k)^2}}\, g^{\mu\nu}(d-1)$$
$$= -g^2 C_2(G)\, \delta^{ab} \int \frac{d^d\ell}{(2\pi)^4} \int dx \frac{1}{(\ell^2-\Delta)^2}\, g^{\mu\nu}(d-1) \Big/ \ell^2 + (1-x)^2 q^2 \Big/$$
$$= \frac{ig^2}{(4\pi)^{d/2}} C_2(G)\, \delta^{ab} \int dx \frac{1}{\Delta^{2-d/2}}$$
$$\Big[-\Gamma(1-d/2)\, g^{\mu\nu} q^2 \left(\tfrac{1}{2} d(d-1)\, x(1-x)\right) - \Gamma(2-d/2)\, g^{\mu\nu} q^2 (d-1)(1-x)^2 \Big]$$

Still have $\Gamma(1-d/2)$ with uncanceled pole.

In renormalizeable theory, one calculation should only give at most one UV divergence: $\Gamma(2-d/2)$. From counterterm counting.

So, still not gauge invariant.

Need to include ghost loop for gauge invariance.

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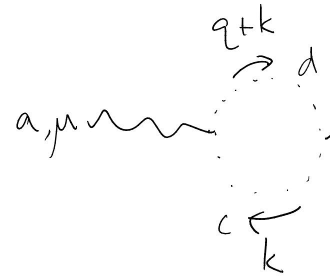


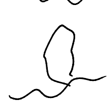
$$i\mathcal{M}_{ghost} = (-1)\int \frac{d^4k}{(2\pi)^4} \cdot \frac{i}{k^2}\,\frac{i}{(k+q)^2}\, g^2 f^{dca}(k+q)^\mu \cdot f^{cbd} k^\nu$$

$$= \frac{ig^2}{(4\pi)^{d/2}} \cdot C_2(G)\,\delta^{ab}\int dx\, \frac{1}{\Delta^{2-d/2}}$$
$$\Big(-\Gamma(1-d/2)\, g^{\mu\nu} q^2\left[\tfrac{1}{2}x(1-x)\right]$$
$$- \Gamma(2-d/2)\, q^\mu q^\nu\,(x(1-x))\Big)$$

Look @ worst divergence, $\Gamma(1-d/2)$: after break.
Return @ 10:10.

Combine terms $\propto \Gamma(1-d/2)$.

$$\frac{-1}{2} g^2 C_2(G)\,\delta^{ab}\int dx\, \frac{1}{(4\pi)^{d/2}}\cdot\frac{1}{(\Delta)^{2-d/2}}\, g^{\mu\nu}\Gamma\left(1-\frac{d}{2}\right) 3(d-1) q^2(x^2-x)$$



$$+ \frac{ig^2}{(4\pi)^{d/2}}\, C_2(G)\,\delta^{ab}\int dx\,\frac{1}{(\Delta)^{2-d/2}}\left(-\Gamma(1-d/2)\, g^{\mu\nu} q^2\left(\tfrac{1}{2}(d)(d-1)x(1-x)\right)\right)$$

$$+ \frac{ig^2}{(4\pi)^{d/2}}\, C_2(G)\,\delta^{ab}\int dx\,\frac{1}{(\Delta)^{2-d/2}}\left(-\Gamma(1-d/2)\, g^{\mu\nu} q^2\left(\tfrac{1}{2}x(1-x)\right)\right)$$

$$= \frac{ig^2 C_2(G)\,\delta^{ab}}{(4\pi)^{d/2}}\int dx\,\frac{1}{(\Delta)^{2-d/2}}\, g^{\mu\nu}\cdot q^2\, x(1-x)\,\Gamma(1-d/2)$$
$$\left[\tfrac{1}{2}\cdot 3(d-1) - \tfrac{1}{2}d(d-1) - \tfrac{1}{2}\right]$$

$$[\ldots] = \frac{3d}{2} - \frac{3}{2} - \frac{d^2}{2} + \frac{d}{2} - \frac{1}{2}$$
$$= \frac{-d^2}{2} + 2d - 2$$
$$= \frac{-1}{2}(d^2 - 4d + 4) = \frac{-1}{2}(d-2)^2$$
$$= (d-2)\left(1-\frac{d}{2}\right)$$

$$= ig^2 C_2(G)\,\delta^{ab} \ldots g^{\mu\nu} q^2 x(1-x)(d-2)\,\Gamma(2-d/2)$$

$$(d-2)\left(1-\tfrac{d}{2}\right)$$

$$= \frac{ig^2 C_2(G)}{(4\pi)^{d/2}} \delta^{ab} \int dx \, \frac{1}{(\Delta)^{2-d/2}} \, g^{\mu\nu} q^2 x(1-x)(d-2)\,\Gamma(2-d/2)$$

Only apparent when including gauge + ghost diagrams.

Remark: Fermions already had $\Gamma(2-d/2)$ pole.

Combine rest:

(M_3)

$$-\frac{1}{2} g^2 C_2(G)\,\delta^{ab} \int dx \, \frac{i}{(4\pi)^{d/2}} \left(\frac{1}{\Delta}\right)^{2-d/2}$$
$$\Big(g^{\mu\nu} q^2 (-5+2x-2x^2)\,\Gamma(2-d/2)$$
$$+ q^\mu q^\nu \big(d(-1+4x-4x^2) + 6(1-x+x^2)\big)\,\Gamma(2-d/2)\Big)$$

(M_4):

$$+ \frac{ig^2}{(4\pi)^{d/2}} C_2(G)\,\delta^{ab} \int dx \, \frac{1}{(\Delta)^{2-d/2}}$$
$$\left(-\Gamma(2-d/2)\, g^{\mu\nu} q^2 (d-1)(1-x)^2\right)$$

(ghost)

$$+ \frac{ig^2}{(4\pi)^{d/2}} C_2(G)\,\delta^{ab} \int dx \left(\frac{1}{\Delta}\right)^{2-d/2}$$
$$\left(\Gamma(2-d/2)\, q^\mu q^\nu \left(x(1-x)\right)\right)$$

$(\Gamma(1-d/2))$

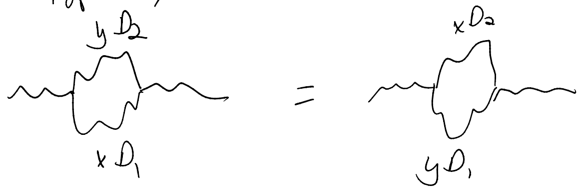
$$+ \frac{ig^2}{(4\pi)^{d/2}} C_2(G)\,\delta^{ab} \int dx \left(\frac{1}{\Delta}\right)^{2-d/2} g^{\mu\nu} q^2 x(1-x)(d-2)\,\Gamma(2-d/2)$$

$$= \frac{ig^2}{(4\pi)^{d/2}} C_2(G)\,\delta^{ab} \int dx \left(\frac{1}{\Delta}\right)^{2-d/2} \Gamma(2-d/2)$$
$$\Big[(g^{\mu\nu} q^2)\left(-\tfrac{1}{2}(-5+2x-2x^2) - (d-1)(1-x)^2 + x(1-x)(d-2)\right)$$
$$+ (q^\mu q^\nu)\left(-\tfrac{1}{2}\big(d(-1+4x-4x^2) + 6(1-x+x^2)\big) + x(1-x)\right)\Big]$$

Non-trivial check:
match coefficients to make total is prop. to $(q^2 g^{\mu\nu} - q^\mu q^\nu)$
in order to satisfy Ward identity.

Not manifest because coefficients only have to match after $\int dx$. Trick is $\int dx \left(\frac{1}{\Delta}\right)^{2-d/2} f(x)$ is symmetric

$\int dx$. Trick is $\int dx \left(\frac{1}{\Delta}\right)^{2-d/2} f(x)$ is symmetric for $x \to 1-x$. True explicitly. Also because



Write $x \to \frac{1}{2}x + \frac{1}{2}(1-x) = \frac{1}{2}$.

$$g^{\mu\nu}q^2 : \quad \frac{5}{2} - x + x^2 - (d-1)(1-2x+x^2) + (x-x^2)(d-2)$$

$$= \frac{5}{2} - \frac{1}{2} + x^2 - (d-1)(1-1+x^2) + \left(\frac{1}{2} - x^2\right)(d-2)$$

$$= 1 + \frac{d}{2} + 4x^2 - 2dx^2$$

$$q^\mu q^\nu : \quad -\frac{d}{2} + 2x^2 d - 1 - 4x^2$$

$$i\mathcal{M}_{\substack{\text{gauge+}\\ \text{ghost}}} = \frac{ig^2}{(4\pi)^{d/2}} C_2(G)\,\delta^{ab} \int_0^1 dx \frac{\Gamma(2-d/2)}{\Delta^{2-d/2}} \left(g^{\mu\nu}q^2 - q^{\mu\nu}\right) \cdot \left(1 + \frac{d}{2} + (4-2d)x^2\right)$$

$$A = \int dx \frac{x}{(\Delta)^{2-d/2}} = \int dx \frac{(1-x)}{(\Delta)^{2-d/2}}$$

$$A = C = B - C$$

$$2C = B$$

$$C = \frac{B}{2}$$

$$\int dx \frac{x}{\Delta^{2-d/2}} = \int dx \frac{\left(\frac{1}{2}\right)}{(\Delta)^{2-d/2}}$$

$$i\mathcal{M} = ig^2 \quad C_2(G)\,\delta^{ab} \int^1 dx \quad \Gamma(2-d/2) \quad (g^{\mu\nu}q^2 - q^{\mu\nu})$$

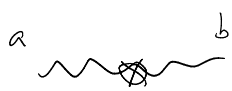
$$i\mathcal{M}_{\substack{\text{gauge+}\\\text{ghost}}} = \frac{ig^2}{(4\pi)^{d/2}} C_2(G)\,\delta^{ab} \int_0^1 dx \, \frac{\Gamma(2-d/2)}{\Delta^{2-d/2}} \left(g^{\mu\nu} q^2 - q^{\mu\nu}\right)$$

$$\cdot \left(1 + \frac{d}{2} + (4-2d)x^2\right)$$

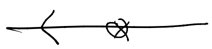
$$i\mathcal{M}_f = \sum_f \frac{ig^2}{16\pi^2} \left(q^2 g^{\mu\nu} - q^\mu q^\nu\right) \delta^{ab} \left(-\frac{8}{3\epsilon} C(r_f)\right)$$

Refer to Section 16.5 in P+S for explicit discussions.

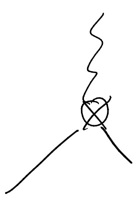
Counterterms + renormalized pert. theory analogous to QED.



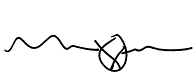
$\delta_3 \, (-i) \left(k^2 g^{\mu\nu} - k^\mu k^\nu\right) \delta^{ab}$



$i\not{p}\, \delta_2$



$i g t^a \gamma^\mu \delta_1$

For , we get δ_3 to cancel $\Gamma(2-d/2)$ divergence.

$$\delta_3 = \frac{g^2}{(4\pi)^2} \frac{\Gamma(2-d/2)}{(M^2)^{2-d/2}} \left(\frac{5}{3} C_2(G) - \frac{4}{3} n_f C(r_f)\right)$$

because $\int dx \left(1 + \frac{d}{2} + (4-2d)x^2\right) \left(\frac{2}{\epsilon}\right)$

$$= \int dx \left(1 + \frac{4-\epsilon}{2} + (4-8+2\epsilon)x^2\right)\left(\frac{2}{\epsilon}\right)$$

$$= \int dx \left(3 - \frac{\epsilon}{2} - 4x^2 + 2\epsilon x^2\right)\left(\frac{2}{\epsilon}\right)$$

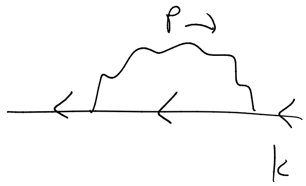
Keep $\frac{1}{\epsilon}$, discard finite in δ_3

$$\int dx \left(3 - \frac{\epsilon}{2} - 4x^2 + 2\epsilon x^2\right)\left(\frac{\alpha}{\epsilon}\right)$$

Keep $\frac{1}{\epsilon}$, discard finite in δ_3

$$= 3x - \frac{4}{3}x^3 \Big|_0^1 \left(\frac{2}{\epsilon}\right) = \frac{5}{3}\left(\frac{2}{\epsilon}\right)$$

δ_2 calculation:

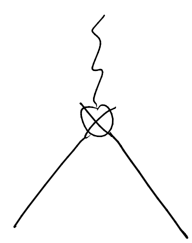
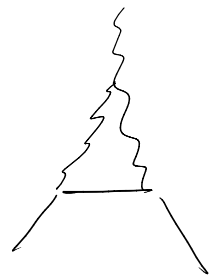
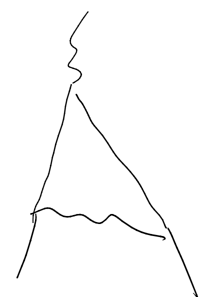


$$= \int \frac{d^4p}{(2\pi)^4} (ig)^2 \gamma^\mu t^a \frac{i(\not p + \not k)}{(p+k)^2} \gamma_\mu t^a \frac{-i}{p^2}$$

$$= \frac{ig^2}{(4\pi)^{d/2}} \cdot C_2(r) \not k \int_0^1 dx\, (1-x)(d-2) \frac{\Gamma(2-d/2)}{\Delta^{2-d/2}}$$

$$\delta_2 = \frac{-g^2}{(4\pi)^2} \cdot \frac{\Gamma(2-d/2)}{(M^2)^{2-d/2}} C_2(r_f)$$

Vertex δ_1 calculation:

 need  and  diagrams.

Get $\delta_1 = \frac{-g^2}{(4\pi)^2} \cdot \frac{\Gamma(2-d/2)}{(M^2)^{2-d/2}} \left(C_2(r_f) + C_2(G) \right)$

Finally, $\beta(g) = \mu \frac{\partial}{\partial\mu}\left(-e\delta_1 + e\delta_2 + \frac{e}{2}\delta_3\right)$ for QED

$$= \mu \frac{\partial}{\partial \mu}\left(-g\delta_1 + g\delta_2 + \frac{g}{2}\delta_3\right) \text{ for Y-M}$$

Note $\delta_1 \neq \delta_2$ in Y-M, self-interaction diagram.

Exercise

$$\beta(g) = \frac{-g^3}{(4\pi)^2}\left(\frac{11}{3} C_2(G) - \frac{4}{3} N_f \, C(r_f)\right)$$

$\hookrightarrow$ # of flavors

1-loop β-fcn. in QCD. $C_2(SU(3)) = 3$

$\hookrightarrow$ # of flavors

1-loop β-fcn. in QCD. $\quad C_2(SU(3)) = 3$

$C(\square) = \frac{1}{2}$.

For small N_f, $\beta < 0$.

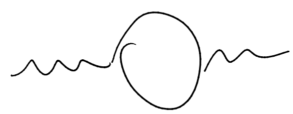
Asymptotic freedom since gauge coupling decreases as we go to higher energy.

Gauge coupling goes strong at low energies: confinement. For QCD $\sim \Lambda_{QCD} \sim 200$ MeV.

Generic for Y-M theories for small N_f.

Next time: More QCD,
SM, Higgs mechanism + Goldstone theorem.

Each fermion:



$$\sum_f \mathrm{Tr}\left(t^b_f t^a_f\right)$$

$$\Rightarrow N_{f_1} \mathrm{Tr}\left[t^b_{f_1} t^a_{f_1}\right] + N_{f_2} \mathrm{Tr}\left[t^b_{f_2} t^a_{f_2}\right]$$

$$= N_{f_1} C(r_{f_1})\delta^{ab} + N_{f_2} C(r_{f_2})\delta^{ab}$$