Lecture 13, June 17, 2020

Wednesday, June 17, 2020 2:07 PM

Last time: Non-Abelian gauge theory + introduction of Fadeev-Popov ghosts.

Today: Two calculations to demonstrate necessity of ghosts

Q: Office hours this week?

Friday, 9a.m. (s.t.) on Zoom

Main takeaway for ghosts:

Ghosts cancel unphysical gauge field dofs. in loop calculations.

Ghosts have wrong spin-statistics for reps. of Poincaré/Lorentz symmetry as external states.

Hence, not physical states/particles.

Instead, think of ghosts as a calculational tool.

Two main calculations

① $f\bar{f} \to G^a G^b$

② 2-pt. correlation fcn. for two gluons, $\Pi_{\mu\nu}(q^2)$.

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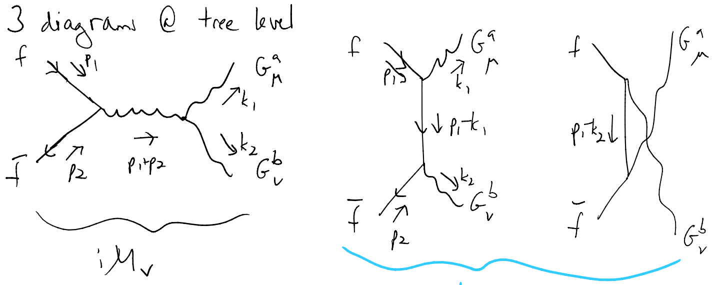
① $f\bar{f} \to G^a G^b$

3 diagrams @ tree level



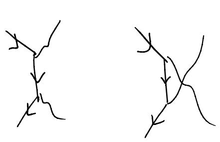
$i\mathcal{M}_v$ $\qquad$ $i\mathcal{M}_f$

$$i\mathcal{M}_f = \bar{v}(p_2)\,(ig t^b \gamma^\nu)\, \frac{i(\not{p}_1-\not{k}_1+m)}{(p_1-k_1)^2-m^2}\, ig t^a \gamma^\mu u(p_1) \cdot \epsilon^*_\mu(k_1)\,\epsilon^*_\nu(k_2)$$
$$+\ \bar{v}(p_2)\,(ig t^a\gamma^\mu)\,\frac{i(\not{p}_1-\not{k}_2+m)}{(p_1-k_2)^2-m^2}\, ig t^b\gamma^\nu u(p_1)\,\epsilon^*_\mu(k_1)\,\epsilon^*_\nu(k_2)$$

Study Ward identity: $\epsilon^*_\nu(k_2) \to k_{2\nu}$.

$$= (-ig^2)\,\epsilon^*_\mu(k_1)\left\{\bar{v}(p_2)\left[(t^b t^a)\,\frac{\not{k}_2(\not{p}_1-\not{k}_1+m)\gamma^\mu}{(p_1-k_1)^2-m^2} + (t^a t^b)\,\frac{\gamma^\mu(\not{p}_1-\not{k}_2+m)\not{k}_2}{(p_1-k_2)^2-m^2}\right]u(p_1)\right\}$$

$\mathcal{M}_{U(1)} =$



These diagrams would be same whether U(1) or SU(N) theory.

$\mathcal{M}_{SU(N)} \overset{\text{rescale group factor}}{\propto} \mathcal{M}_{U(1)}$

The SU(N) generators live in a diff. space + act on different indices, compared to Lorentz indices.

Note: $p_1+p_2=k_1+k_2 \Rightarrow p_1-k_1=k_2-p_2$

$\bar v(p_2)(\not p_2+m)=0$

$(\not p_1-m)u(p_1)=0$

$$=(-ig^2)\,\epsilon^*_\mu(k_1)\left\{\bar v(p_2)\left[(t^bt^a)\,\frac{(\not k_2-\not p_2-m)(\not p_1-\not k_1+m)}{(p_1-k_1)^2-m^2}\gamma^\mu + (t^at^b)\,\gamma^\mu\frac{(\not p_1-\not k_2+m)(\not k_2-\not p_1+m)}{(p_1-k_2)^2-m^2}\right]u(p_1)\right\}$$

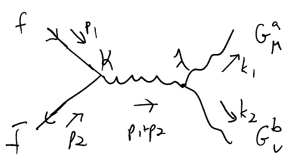
$$=(-ig^2)\,\epsilon^*_\mu(k_1)\left\{\bar v(p_2)\left[t^bt^a\,\frac{\cancel{(\not p_1-\not k_1-m)}(\not p_1-\not k_1+m)}{\cancel{(p_1-k_1)^2-m^2}}\gamma^\mu - t^at^b\,\gamma^\mu\frac{\cancel{(\not p_1-\not k_2+m)}(\not p_1-\not k_2-m)}{\cancel{(p_1-k_2)^2-m^2}}\right]u(p_1)\right\}$$

$$=(-ig^2)\,\epsilon^*_\mu(k_1)\,\bar v(p_2)\,[t^b,t^a]\,\gamma^\mu u(p_1)$$

$$=ig^2\,\epsilon^*_\mu(k_1)\,\bar v(p_2)\,[t^a,t^b]\,\gamma^\mu u(p_1)$$

$$=-g^2 f^{abc}t^c\,\epsilon^*_\mu(k_1)\,\bar v(p_2)\gamma^\mu u(p_1)$$

Gauge diagram (set $\xi=1$).



$i\mathcal{M}_V$

$$i\mathcal{M}_V=\bar v(p_2)\,(ig t^c\gamma^\kappa)\,u(p_1)\,\epsilon^*_\mu(k_1)\,\epsilon^*_\nu(k_2)\;\frac{-i\,g_{\kappa\lambda}}{(p_1+p_2)^2}\cdot g f^{abc}\left[g^{\mu\nu}(-k_1+k_2)^\lambda+g^{\nu\lambda}(-k_2-p_1-p_2)^\mu + g^{\lambda\mu}(p_1+p_2+k_1)^\nu\right]$$

Again, $\epsilon^*_\nu(k_2)\to k_{2\nu}$.

$$=\frac{g^2f^{abc}t^c}{(p_1+p_2)^2}\,\bar v(p_2)\gamma^\kappa u(p_1)\,\epsilon^*_\mu(k_1)\,g_{\kappa\lambda}\left[k_2^\mu(-k_1+k_2)^\lambda+k_2^\lambda(-k_2-k_1-k_2)^\mu+g^{\mu\lambda}(2k_1+k_2)\cdot k_2\right]$$

$$=\frac{g^2f^{abc}t^c}{(p_1+p_2)^2}\,\bar v(p_2)\gamma^\kappa u(p_1)\,\epsilon^*_\mu(k_1)\,g_{\kappa\lambda}\left[-k_2^\mu k_1^\lambda+k_2^\mu k_2^\lambda-2k_2^\mu k_2^\lambda-k_1^\mu k_2^\lambda+2(k_1\cdot k_2)g^{\mu\lambda}\right]$$

$$= \frac{g^2 f^{abc} t^c}{(p_1+p_2)^2}\Big[\Big(\bar v(p_2)\,(\not k_1+\not k_2)\,u(p_1)\,(-k_2\cdot\epsilon^*(k_1))\Big)$$
$$-\bar v(p_2)\,\not k_2\,u(p_1)\;k_1\cdot\epsilon^*(k_1)$$
$$+\;2(k_1\cdot k_2)\,\bar v(p_2)\,\gamma^\mu\,u(p_1)\Big]$$

Note $(p_1+p_2)^2 = (k_1+k_2)^2 = 2k_1\cdot k_2$

$$\bar v(p_2)\,(\not k_1+\not k_2)\,u(p_1) = \bar v(p_2)\,(\not p_1 - m + \not p_2 + m)\,u(p_1)$$
$$= 0.$$

Third line cancels $i\mathcal{M}_f$.

Total $i\mathcal{M}_{tot} = i\mathcal{M}_f + i\mathcal{M}_v$

$$= \frac{-g^2 f^{abc} t^c}{(p_1+p_2)^2}\,\bar v(p_2)\,\not k_2\,u(p_1)\;k_1\cdot\epsilon^*(k_1)$$

For on-shell gauge boson, ϵ is transverse, $k_1\cdot\epsilon^*(k_1)=0$.

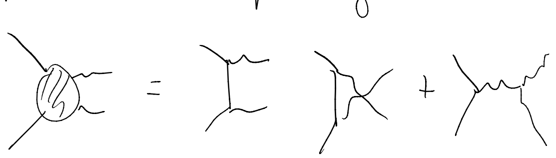
Unsatisfactory here!

Ward identity should be valid off-shell;

embed tree diagram in a loop diagram

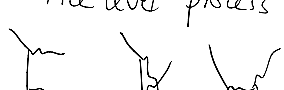
In this four fermion diagram, the internal gauge propagators are not on-shell + we would lose Ward identity of overall diagram. Direct connection to optical theorem, cut along vertical dotted line + replace cut propagator by $-ig_{\mu\nu}\,(-2\pi i\,\delta(k_i^2))$, get non-zero $\mathrm{Im}(\mathcal{M})$.

But no corresponding tree-level process!



Optical thm.

related to tree-level process above



$\mathcal{M} \propto k\cdot\epsilon^*(k_1) = 0$ on-shell above 

$\mathcal{M} \propto \quad k_i \cdot \epsilon^*(k_i) = 0$ on-shell, ϵ is transverse.

Loop diagram does not have this condition.

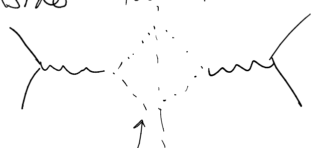
Virtual gluons give a nonzero contribution that violate gauge invariance.

They have no on-shell, tree-level corresponding diagram.

Ghosts fix this problem!

Break until 3:14pm.

Reconsider four fermion @ 1-loop.



ghosts

At tree-level

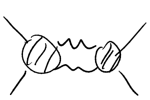
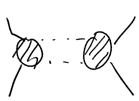
$$i\mathcal{M}_{ghost} = \bar{v}(p_2)\, i g t^c \gamma^\kappa u(p_1) \left(\frac{-i g_{\kappa\lambda}}{(p_1+p_2)^2} \right) \left(-g f^{acb} k_1^\lambda \right)$$

$$= \bar{v}(p_2)\, g^2 \frac{f^{abc} t^c}{(p_1+p_2)^2} \gamma^\kappa u(p_1)\, k_1^\kappa$$

Matching what is needed:

one diagram for ghosts on k_1 + k_2 legs.

In the end, sum over

 +  only leave physical gauge boson states in a cut diagram.

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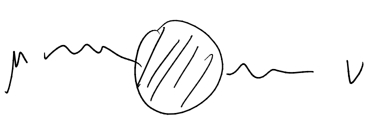
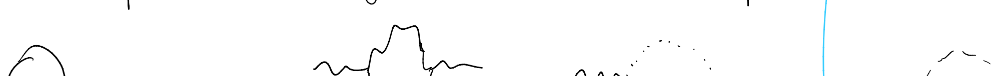
② 2pt. correlation fcn. $\Pi_{\mu\nu}(q^2)$



Resolved @ 1-loop:

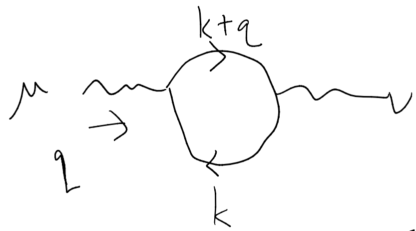
fermion loop	Gauge	Ghost loop	In principle, can add Scalar loops



fermion loop Gauge ghost loop scalar loops

$$\mathcal{L} \supset \left(|D_\mu \Phi|^2 \right) + \bar{\psi} i \not{D} \psi + \frac{-1}{4} F^a_{\mu\nu} F^{a\mu\nu} + \text{ghosts}$$

Fermion calc.



amputate $\epsilon_\mu \epsilon_\nu^*$

$$i\mathcal{M} = (-1) \int \frac{d^4k}{(2\pi)^4} \mathrm{Tr}\left[\frac{i(\not{k}+m)}{k^2-m^2} \, i g t^b \gamma^\nu \, \frac{i(\not{k}+\not{q}+m)}{(k+q)^2-m^2} \, i g t^a \gamma^\mu \right]$$

$$= -g^2 \, \mathrm{Tr}[t^b t^a] \int \frac{d^4k}{(2\pi)^4} \frac{1}{k^2-m^2} \frac{1}{(k+q)^2-m^2}$$
$$\mathrm{Tr}\left[\not{k} \gamma^\nu (\not{k}+\not{q}) \gamma^\mu + m^2 \gamma^\nu \gamma^\mu \right]$$

$$= -g^2 \, \mathrm{Tr}[t^a t^b] \int \frac{d^4k}{(2\pi)^4} \int_0^1 dx \frac{1}{(k^2 + 2k\cdot q x - m^2 + xq^2)^2}$$
$$4\left(k^\nu (k+q)^\mu - g^{\mu\nu} k\cdot(k+q) + k^\mu (k+q)^\nu + m^2 g^{\mu\nu} \right)$$

$$\mathrm{Tr}[t^a t^b] = C(r) \delta^{ab}$$

$$= -4 g^2 C(r) \delta^{ab} \int \frac{d^4k}{(2\pi)^4} \int_0^1 dx \frac{1}{(k^2 + 2k\cdot q x - m^2 + xq^2)^2}$$
$$\left(2k^\mu k^\nu + q^\mu k^\nu + k^\mu q^\nu + g^{\mu\nu}(-k^2 - k\cdot q + m^2) \right)$$

$$[t^a, t^b] = i f^{abc} t^c$$

Shift $\quad l \equiv k + xq$

Denominator: $\quad l^2 - m^2 + xq^2 - x^2q^2$

$\Delta \equiv m^2 - xq^2 + x^2q^2$

Perform shift in numerator:

$$\star = \Big(2(l-xq)^\mu (l-xq)^\nu + q^\mu (l-xq)^\nu + (l-xq)^\mu q^\nu$$
$$+ g^{\mu\nu}\left(-(l-xq)^2 - (l-xq)\cdot q + m^2\right) \Big)$$

Drop linear l terms:

$$\star = 2l^\mu l^\nu + 2x^2 q^\mu q^\nu - xq^\mu q^\nu - xq^\mu q^\nu$$
$$+ g^{\mu\nu}\left(-l^2 - x^2q^2 + xq^2 + m^2\right)$$
$$= 2l^\mu l^\nu + 2q^\mu q^\nu (x^2 - x) - l^2 g^{\mu\nu}$$
$$+ g^{\mu\nu}\left(-x^2q^2 + xq^2 + m^2\right)$$

Use dim. reg.

$$\int \frac{d^d l}{(2\pi)^d} \frac{2l^\mu l^\nu - l^2 g^{\mu\nu}}{(l^2-\Delta)^2} = \int \frac{d^d l}{(2\pi)^d} \frac{\left(2\cdot\frac{1}{d}l^2 - l^2\right) g^{\mu\nu}}{(l^2-\Delta)^2}$$
$$= \left(\frac{2}{d} - 1\right) g^{\mu\nu} \int \frac{d^d l}{(2\pi)^d} \frac{l^2}{(l^2-\Delta)^2}$$
$$= \left(\frac{2}{d} - 1\right) g^{\mu\nu} \left[\frac{-1 \cdot i}{(4\pi)^{d/2}} \cdot \left(\frac{d}{2}\right) \frac{\Gamma(1-\frac{d}{2})}{\Gamma(2)} \left(\frac{1}{\Delta}\right)^{1-\frac{d}{2}} \right]$$
$$= \frac{-i g^{\mu\nu}}{(4\pi)^{d/2}} \frac{\Gamma(2-\frac{d}{2})\ \Delta}{(\Delta)^{2-d/2}}$$

$$\int \frac{d^d l}{(2\pi)^d} \frac{1}{(l^2-\Delta)^2} = \frac{i}{(4\pi)^{d/2}} \cdot \frac{\Gamma(2-\frac{d}{2})}{(\Delta)^{2-d/2}}$$

Combine:

$$= -4q^2 C(r)\, \delta^{ab} \int_0^1 dx$$

combine:

$$= -4g^2 C(r)\delta^{ab}\int_0^1 dx$$

$$\Bigg[\frac{-ig^{\mu\nu}}{(4\pi)^{d/2}} \cdot \frac{\Gamma(2-\frac{d}{2})}{\Delta^{2-d/2}} \cdot \Delta$$

$\Delta \equiv m^2 - xq^2 + x^2q^2$

$$+ \left(2q^\mu q^\nu (x^2-x) + g^{\mu\nu}(-x^2q^2 + xq^2 + m^2)\right) \cdot \frac{i}{(4\pi)^{d/2}} \frac{\Gamma(2-\frac{d}{2})}{\Delta^{2-d/2}}\Bigg]$$

$$= -4g^2 C(r)\delta^{ab}\int_0^1 dx$$

$$\left[\frac{i}{(4\pi)^{d/2}} \frac{\Gamma(2-\frac{d}{2})}{\Delta^{2-d/2}}\right]$$

$$\left[-g^{\mu\nu}(\cancel{m^2} - xq^2 + x^2q^2) + g^{\mu\nu}(-x^2q^2 + xq^2 + \cancel{m^2}) + 2q^\mu q^\nu (x^2-x)\right]$$

$$= -4g^2 C(r)\delta^{ab}\int_0^1 dx \left[\frac{i}{(4\pi)^{d/2}} \frac{\Gamma(2-d/2)}{\Delta^{2-d/2}}\right]$$

$$\left(q^2 g^{\mu\nu} \cdot 2x(1-x) + q^\mu q^\nu 2x(x-1)\right)$$

$$= -4g^2 C(r)\delta^{ab}\int_0^1 dx \left[\frac{i}{(4\pi)^{d/2}} \frac{\Gamma(2-d/2)}{\Delta^{2-d/2}}\right]$$

$$2x(1-x)\left(q^2 g^{\mu\nu} - q^\mu q^\nu\right)$$

$q^2 g^{\mu\nu} - q^\mu q^\nu$ is guaranteed by Ward identity.

$$= ig^2 C(r)\delta^{ab}(q^2 g^{\mu\nu} - q^\mu q^\nu)\int_0^1 dx \frac{8x(x-1)}{(4\pi)^{d/2}} \frac{\Gamma(2-d/2)}{\Delta^{2-d/2}}$$

For divergent piece, $d = 4-\epsilon$

$$\frac{1}{(4\pi)^{d/2}} \frac{\Gamma(2-d/2)}{\Delta^{2-d/2}} \to \frac{1}{16\pi^2} \cdot \frac{2}{\epsilon}$$

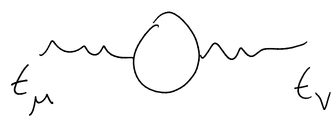
$$\int_0^1 dx\, 8x^2 - 8x = \frac{8}{3} - 4 = \frac{-4}{3}$$

$$= \frac{ig^2 C(r)\delta^{ab}}{16\pi^2}(q^2 g^{\mu\nu} - q^\mu q^\nu)\left(-\frac{4}{3}\cdot\frac{2}{\epsilon}\right)$$

$$= \frac{ig^2 C(r)}{16\pi^2} \delta^{ab} (q^2 g^{\mu\nu} - q^\mu q^\nu) \left(-\frac{4}{3} \cdot \frac{2}{\epsilon} \right)$$

For N_F flavors

$$= \sum_f \frac{ig^2}{16\pi^2} (q^2 g^{\mu\nu} - q^\mu q^\nu) \delta^{ab} \left(\frac{-8}{3\epsilon} \cdot C(r_f) \right)$$



$$i\mathcal{M} = \epsilon^a_\mu \, \epsilon^{*b}_\nu \, \Pi^{(ab)(\mu\nu)}(q^2)$$

$$A_\mu = A^a_\mu \, t^a$$

adjoint rep.

$$\epsilon_\mu = \epsilon^a_\mu \, t^a(r_f)$$

adjoint representation matrix