Lecture 12, June 10, 2020

Wednesday, June 10, 2020 2:06 PM

Last time: Group theory + Lie algebras

Today: Non-Abelian Gauge theory

Quantization of Abelian theory with fcn'al method

↳ Choice of ξ-gauge

Quantization of non-Abelian theory with fcn'al method

HW 4 due June 15

Terminology: ① Yang-Mills theory

$SU(N)$ gauge theory w/ no matter content (so no fermions)

② QCD / QCD-like

$SU(N)$ gauge theory w/ N_f vectorlike fermions

$N=3$, $N_f = 2$ or 3 for our QCD.

$$\mathcal{L} = -\frac{1}{4}(G^a_{\mu\nu})^2 + \bar{\Psi}(i\not{D} - m)\Psi$$

$$D_\mu = \partial_\mu - ig A^a_\mu t^a$$

$$G^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^{abc} A^b_\mu A^c_\nu$$

a, b, c are color indices

t^a are set of generators for Lie algebra of $SU(N)$.

Ψ is in fundamental rep. of $SU(N)$.

☆ Extra piece in $G^a_{\mu\nu} \ni g f^{abc} A^b_\mu A^c_\nu$

In contrast to Abelian field strength

Extra piece includes one power of coupling const.

+ also quadratic in field.

Fermion propagator

$$\frac{i\,\delta_{ij}}{\not{k} - m} \quad \text{for } \langle \Psi_i \bar{\Psi}_j \rangle$$

color indices

Fcn'al method for 2pt. correlation fcn.

Massless vector:

$$\frac{-i g_{\mu\nu}}{k^2}\,\delta^{ab} \quad \text{for } \langle A^a_\mu A^b_\nu \rangle$$

Interacting Lagrangian:

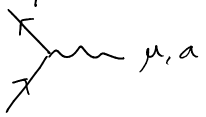
$$\mathcal{L} = -\frac{1}{4}(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^{abc} A^b_\mu A^c_\nu)^2 + \bar{\Psi}(i\not{\partial} - m)\Psi$$

$$= \mathcal{L}_{free} + g A^a_\lambda \bar{\Psi} \gamma^\lambda t^a \Psi$$

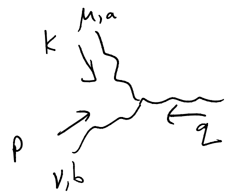
$$- g f^{abc}(\partial_\kappa A^a_\lambda) A^{\kappa b} A^{\lambda c}$$

$$= \mathcal{L}_{free} + g A^a_\lambda \bar{\Psi} \gamma^\lambda t^a \psi$$
$$- g f^{abc} (\partial_\kappa A^a_\lambda) A^{\kappa b} A^{\lambda c}$$
$$- \frac{1}{4} g^2 (f^{eab} A^b_\kappa A^c_\lambda)(f^{ecd} A^{\kappa c} A^{\lambda d})$$

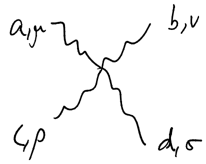
Can easily derive



$i g t^a \gamma^\mu$



$g f^{abc} \left[g^{\mu\nu}(k-p)^\lambda + g^{\nu\lambda}(p-q)^\mu + g^{\lambda\mu}(q-k)^\nu \right]$



$$-ig^2 \Big[f^{abe} f^{cde} (g^{\mu\rho} g^{\nu\sigma} - g^{\mu\sigma} g^{\nu\rho})$$
$$+ f^{ace} f^{bde} (g^{\mu\nu} g^{\rho\sigma} - g^{\mu\sigma} g^{\nu\rho})$$
$$+ f^{ade} f^{bce} (g^{\mu\nu} g^{\rho\sigma} - g^{\mu\rho} g^{\nu\sigma}) \Big]$$

Note: different power of same gauge coupling in 3 pt. & 4 pt. vertices. Critical for overall gauge invariance. In general, must draw <u>all</u> diagrams of a given $O(g^n)$ to get gauge invariant result. & # of loops

4 gluon scattering:

All come with $\mathcal{M} \propto g^2$.

Quantize Abelian theory:

Recall the need for gauge fixing.

Abelian $\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \bar{\Psi} i \not{D} \Psi$

or $-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + J_\mu A^\mu$

Then, EOM for A_μ is

$$(k^2 g_{\mu\nu} - k_\mu k_\nu) A_\nu = J_\mu$$

Since $S = \int d^4x \left(\frac{-1}{4} F_{\mu\nu} F^{\mu\nu} \right) + \ldots$

$$= \frac{1}{2} \int d^4x \, A_\mu(x) \left(\partial^2 g_{\mu\nu} - \partial_\mu \partial_\nu \right) A_\nu(x)$$

$$= \frac{1}{2} \int \frac{d^4k}{(2\pi)^4} \tilde{A}_\mu(k) \left(-k^2 g^{\mu\nu} + k^\mu k^\nu \right) \tilde{A}_\nu(k)$$

Not invertible because operator $(-k^2 g^{\mu\nu} + k^\mu k^\nu)$ has det $= 0$ from k_μ eigenvector. Cannot uniquely solve for A_μ given J_μ, because $A_\mu \to A_\mu + \frac{1}{e}\partial_\mu \alpha$, then shifted A_μ obeys same EOM. In the functional integral

$$\int \mathcal{D}A \, e^{iS[A]} \equiv \int \mathcal{D}A^0 \mathcal{D}A^1 \mathcal{D}A^2 \mathcal{D}A^3 \, e^{iS[A]},$$

the field configurations that are gauge-equivalent to 0 gives an infinite redundancy. Goal is to modify $S[A]$ to calculate + cancel redundancy in observables.

$\Rightarrow$ Fadeev-Popov method.

Consider $G(A) = \partial^\mu A_\mu(x) - w(x)$, $w(x)$ is any scalar fcn. Then, $G(A^\alpha) = G(A_\mu + \frac{1}{e}\partial_\mu \alpha)$

$$= \partial^\mu A_\mu(x) + \frac{1}{e}\partial^2 \alpha - w(x)$$

[If we have Lorentz gauge, $w(x) = 0$

$$G(A^\alpha) = \partial^\mu A_\mu + \frac{1}{e}\partial^2 \alpha \,]$$

We can introduce a functional δ-fcn, $\delta(G(A))$ to constrain final integral to cover only configurations with $G(A) = 0$.

Identity: $$1 = \int \mathcal{D}\alpha(x) \, \delta(G(A^\alpha)) \underbrace{\det\left(\frac{\delta G(A^\alpha)}{\delta \alpha} \right)}_{\text{infinite product}}$$

↳ infinite product of δ-fcns for each pt. x.

From $1 = \left(\prod_i \int da_i\right) \delta^{(n)}(g(a)) \det\left(\frac{\partial g_i}{\partial a_j}\right)$

Recall $$\int_{-\infty}^{\infty} f(x)\, \delta(g(x))\, dx = \sum_i \frac{f(x_i)}{|g'(x_i)|}$$ for x_i roots of $g(x)$

↳ This is one-variable version of above.

The $\frac{1}{|g'(x_i)|}$ comes from Jacobian for changing arg. x to $g(x)$.

Analogously, changing a_i to $g(a)$ requires Jacobian $\det\left(\frac{\partial g_i}{\partial a_j}\right)$. Original identity is continuum limit where $\prod_i \int da_i \to \int D\alpha$.

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Use identity inside $\int DA\, e^{iS[A]}$

$$\star = \int DA\, D\alpha\, \delta(G(A^\alpha)) \det\left(\frac{\delta G(A^\alpha)}{\delta \alpha}\right) e^{iS[A]}$$

Remark: $\det\left(\frac{\delta G(A^\alpha)}{\delta \alpha}\right) = \det\left(\frac{\partial^2}{e}\right)$ is ind. of α and A.

Secondly: $S[A]$ is gauge invariant, change variables to $S[A^\alpha]$, then $DA = DA^\alpha$, rename A^α back to A, removes α dependence.

$$\star = \det\left(\frac{\delta G(A^\alpha)}{\delta \alpha}\right)\left(\int D\alpha\right) \int DA\, e^{iS[A]}\, \delta(G(A))$$

so functional integral $\int DA$ restricted to physically inequivalent field configurations where $G(A) = 0$.

So functional integral $\int \mathcal{D}A$ restricted to physically inequivalent field configurations where $G(A)=0$.
If $G(A) \neq 0$, integral vanishes.

Next, $G(A) = \partial_\mu A^\mu - w$

$$\int \mathcal{D}A\, e^{iS[A]} = \det\left(\frac{1}{e}\partial^2\right) \left(\int \mathcal{D}\alpha\right) \int \mathcal{D}A\, e^{iS[A]}\, \delta(\partial^\mu A_\mu - w(x))$$

for any w. So, linearly combine different w with appropriate normalization. Introduce integration over w:

$$N(\xi) \int \mathcal{D}w\, \exp\left[-i\int d^4x\, \frac{w^2}{2\xi}\right] \det\left[\frac{\partial^2}{e}\right]\left(\int \mathcal{D}\alpha\right)$$
$$\int \mathcal{D}A\, e^{iS[A]}\, \delta(\partial^\mu A_\mu - w)$$
$$= \underbrace{N(\xi) \det\left(\frac{1}{e}\partial^2\right) \left(\int \mathcal{D}\alpha\right)}_{\substack{\text{prefactor} \\ \text{will cancel in corr. fcns.}}} \cdot \int \mathcal{D}A\, e^{iS[A]} \underbrace{\exp\left[-i\int d^4x\, \frac{1}{2\xi}(\partial^\mu A_\mu)^2\right]}_{\substack{\text{effectively new term} \\ \text{added to } \mathcal{L}.}}$$

$$\langle \Omega | T\, \mathcal{O}(A) | \Omega \rangle$$
$$= \lim_{T \to \infty(1-i\epsilon)} \frac{\int \mathcal{D}A\, \mathcal{O}(A) \exp\left[i\int_{-T}^{T} d^4x\, \mathcal{L}\right]}{\int \mathcal{D}A\, \exp\left[i\int_{-T}^{T} d^4x\, \mathcal{L}\right]}$$

Given $\mathcal{O}(A)$ is gauge invariant,

$$= \lim_{T \to \infty(1-i\epsilon)} \frac{\int \mathcal{D}A\, \mathcal{O}(A) \exp\left[i\int_{-T}^{T} d^4x \left(\mathcal{L} - \frac{1}{2\xi}(\partial^\mu A_\mu)^2\right)\right]}{\int \mathcal{D}A\, \exp\left[i\int_{-T}^{T} d^4x \left(\mathcal{L} - \frac{1}{2\xi}(\partial^\mu A_\mu)^2\right)\right]}$$

Now, Abelian propagator is ξ-dependent.

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Get

$$\mu \overset{\displaystyle\sim\sim\sim}{\underset{k}{}} \nu \qquad \frac{-i}{k^2+i\epsilon}\left(g^{\mu\nu} - (1-\xi)\frac{k^\mu k^\nu}{k^2}\right)$$

Popular choices:

$\xi = 0$ Landau gauge

$\xi = 1$ Feynman gauge.

Now, consider non-Abelian gauge theory.

$\boxed{\text{P\&S 16.2.}}$

In non-Abelian case,

$$1 = \int D\alpha(x)\, \delta(G(A^\alpha))\, \det\left(\frac{\delta G(A^\alpha)}{\delta\alpha}\right)$$

A^α is gauge-transformed field, has α dependence.

finite $\quad (A^\alpha)_\mu^a t^a = e^{i\alpha^a t^a}\left(A_\mu^b t^b + \frac{i}{g}\partial_\mu\right)e^{-i\alpha^c t^c}$

infinitesimal

$$(A^\alpha)^a_\mu = A^a_\mu + \frac{1}{g}\partial_\mu\alpha^a + f^{abc}A^b_\mu\alpha^c$$

$$= A^a_\mu + \frac{1}{g}D_\mu\alpha^a$$, where D_μ is covariant derivative of α^a in adjoint rep.

Follow same steps from before,

but now $\det\left(\frac{\delta G(A^\alpha)}{\delta\alpha}\right)$ is no longer independent of α or A,

and not simply a normalization factor.

$$\det\left(\frac{\delta G(A^\alpha)}{\delta\alpha}\right) = \det\left(\frac{1}{g}\partial^\mu D_\mu\right)$$

Instead, we interpret det as a fcn'al integral over new set of <u>anti-commuting</u> scalar fields that transform in <u>adjoint</u> rep.: "Fadeev-Popov ghosts"

$$\det\left(\frac{1}{g}\partial^\mu D_\mu\right) = \int \mathcal{D}c\,\mathcal{D}\bar{c}\,\exp\left[i\int d^4x\,\bar{c}(-\partial^\mu D_\mu)c\right]$$

(with $\frac{1}{g}$ in normalization of $c, \bar{c}$)

Recall: Grassmann integrals:

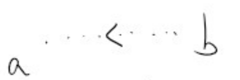
$$\left(\prod_i \int d\theta_i^* d\theta_i\right) e^{-\theta_i^* B_{ij}\theta_j} = \det B$$

So, we can generate this $\det\left(\frac{1}{g}\partial^\mu D_\mu\right)$ from fermionic Grassmann integral.

Thus, quantization of non-Abelian theory requires introduction of Fadeev-Popov ghosts c + anti-ghosts $\bar{c}$.

As before, we get $\frac{1}{2\xi}(\partial^\mu A_\mu)^2$ in $\mathcal{L}$, and

$$\mathcal{L}_{ghost} = \bar{c}^a\left(-\partial^2\delta^{ac} - g\,\partial^\mu f^{abc}A_\mu^b\right)c^c$$



$= \dfrac{i\delta^{ab}}{k^2}$

$-g f^{abc} k^\mu$

Remaining Non-Abelian discussion same as before: shift A to A^α, relabel, Gaussian integrate over ω, get ξ-dependent term in $\mathcal{L}$

$$\frac{-i\delta^{ab}}{k^2+i\epsilon}\left(g^{\mu\nu} - (1-\xi)\frac{k^\mu k^\nu}{k^2}\right)$$

Next lecture: two main calculations

$f\bar{f} \to G^a G^b$

$\Pi_{\mu\nu}(q^2)$ for G.

$\Pi_{\mu\nu}(q^2)$