Lecture 11, June 3, 2020

Wednesday, June 3, 2020 2:09 PM

Last time: Began unit on non-Abelian gauge theory.

Today: Group theory

Reminder: Section for HW3 is Monday, June 8.
HW4 now posted on Reader, due June 15.

First: Q+A from HW3:

Stevenson reading $R(Q) \simeq \frac{\sigma_{tot}(e^+e^- \to \text{hadrons})}{\sigma(e^+e^- \to \mu^+\mu^-)}$

Premise: $\mathcal{L}$ has no mass parameters.

R is dimensionless:

$R \not\propto [\, Q^n,\ n > 0\,]$

$R \propto \left[\left(\frac{Q}{\Lambda}\right)^n,\ n>0\right]$ ✓

↳ what is Λ, where does it come from?

Contradiction because we know $R(Q)$ has non-trivial Q dependence, we need a way to generate Λ from Lagrangian.

Dimensionless $\mathcal{L}$: Has rescaling symmetry.

(Reminiscent of distinction between coordinate + distance:

Recall that we applied a metric to a coordinate system to obtain a distance: $x_\mu x_\nu g^{\mu\nu} = x^2$.

$$\Rightarrow \boxed{\Lambda_\mu{}^\alpha x_\alpha}\,\boxed{\Lambda_\nu{}^\beta x_\beta}\, g^{\mu\nu} = x^2 \,)$$

Same distinction in rescaling of spacetime:

$$S = -\int d^4x\ \mathcal{L}(x)$$

$$x^\mu \to K x^\mu$$

$$S = -\int \frac{d^4x}{K^4}\ \mathcal{L}(K x^\mu)$$

↑ invariant

Assume now $\mathcal{L}$ has no dimensionful couplings.

invariant. Assume now $\mathcal{L}$ has no dimensionful couplings.

The question becomes how/whether does the renormalization μ transforms with κ?

This is why eqn. 25 has

$$\tilde{G}(\kappa x, \kappa\mu, r) = \kappa^{D-\gamma(\mu)} \tilde{G}(x, \mu, r)$$

same as Λ from above

Basic group theory:

Motivated by $\Psi(x) \to V(x)\Psi(x)$

$$V(x) \sim 1 + i\alpha^a(x) t^a + O(\alpha^2)$$

This describes a continuous group, called "Lie algebra"

Groups are sets of elements with a multiplication rule $\cdot$

1. If $x, y \in G$, then $x \cdot y \in G$. (Closure)
2. There is an identity element $e \cdot x = x \cdot e = x$ $\forall x$.
3. For all $x \in G$, inverse exists + $x^{-1} \cdot x = x \cdot x^{-1} = e$
4. Associativity, $a \cdot (b \cdot c) = (a \cdot b) \cdot c$

Representations: Can map set of elements such that algebraic mapping preserves group multiplication table:

$$D(x)\,D(y) = D(x \cdot y).$$

In physics, will use unitary operator on Hilbert space as D.

ex. $D(n) = e^{in\theta}$ for group $\mathbb{Z}$ under addition.

All integers, operation is $+$.

Check: $z_1, z_2 \in \mathbb{Z}$

$$z_1 + z_2 = z_3$$

$$D(z_1)\,D(z_2) = e^{iz_1\theta} e^{iz_2\theta} = e^{i(z_1+z_2)\theta} = D(z_3) \checkmark$$

$$D(z_1)\, D(z_2) = e^{iz_1\theta} e^{iz_2\theta} = e^{i(z_1+z_2)\theta} = D(z_3) \quad \checkmark$$

Distinguish Abelian vs. non-Abelian groups.

Abelian: Group operation is commutative $\forall$ group elements.

non-Abelian: " " is <u>not</u> " " "

Can introduce similarity transformation

$$D_2(x) = S\, D_1(x)\, S^{-1} \quad \forall x \in G.$$

Representation is reducible if we can find a similarity transform s.t. D' is block-diagonal. Then D' is a direct sum, $D' = D_1 \oplus D_2$ with vector space that D' operates on breaking up into invariant subspaces.

Otherwise, if no similarity trans. exists, rep. is irreducible.

If we require group elements labeled by cont. params. s.t. any infinitesimal group element g follows

$$g(\alpha) = 1 + i\alpha^a t^a + O(\alpha^2).$$

α^a: infinitesimal group parameters,

t^a: generators, Hermitian ops.

This type of continuous group is a Lie group.

Return @ 3:10.

Generators span space of infinitesimal group tranformations, so provide a basis to define

$$[t^a, t^b] = i f^{abc} t^c.$$

f^{abc}: structure constants.

Lie Algebra: vector space spanned by t^a + operation of commutation

Jacobi Identity:

$$[t^a,[t^b,t^c]] + [t^b,[t^c,t^a]] + [t^c,[t^a,t^b]] = 0$$

$$[t^a,[t^b,t^c]] + [t^b,[t^c,t^a]] + [t^c,[t^a,t^b]] = 0$$
$$f^{ade} f^{bcd} + f^{bde} f^{cad} + f^{cde} f^{abd} = 0$$

Remark: Global properties of groups can be different even with same Lie algebra.

Focus on compact Lie groups, since transform finite set of fields.

A few more definitions:

Given a general Lie algebra, can separate out all commuting generators $\ni$ called "center" of group. Each generates $U(1)$.

If group has no $U(1)$ factors, algebra is semi-simple.

No nontrivial invariant subgroup: algebra is simple.

Complete taxonomy of all compact simple Lie algebras (Killing + Cartan)

1) $U(N) = SU(N) \times U(1)$

$SU(N)$ = unitary $N \times N$ transformations

$\det U = 1$

$\mathrm{tr}[t^a] = 0$

N^2-1 generators.

2) $O(N) = SO(N) \times$ reflection.

$SO(N)$ = unitary $N\times N$ transformations that also preserve symmetric inner product

aka rotation group in N dimensions.

$\frac{N(N-1)}{2}$ generators.

3) $Sp(N)$: symplectic group.

unitary $N\times N$ matrices that preserve

$\eta_a \epsilon_{ab} \xi_b$, $\epsilon_{ab} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, N even.

(skew-symmetric) $\hookrightarrow \frac{N}{2} \times \frac{N}{2}$ blocks (skew-symmetric) $\hookrightarrow \frac{N}{2} \times \frac{N}{2}$ blocks

$\frac{N(N+1)}{2}$ generators.

4.) Exceptional: G_2, F_4, E_6, E_7, E_8.

Returning to representations:

dimension of rep. = dim. of vector space on which it exists.

Representation matrices change depending on rep.: t_r^a.

Define $D^{ab} \equiv \mathrm{tr}[t_r^a t_r^b] = C(r)\, \delta^{ab}$

$C(r)$ is Casimir invariant of rep. r.

Also, $f^{abc} = \frac{-i}{C(r)} \mathrm{tr}\left\{ [t_r^a, t_r^b], t_r^c \right\}$

so f^{abc} is totally antisymmetric in a,b,c.

Given rep. r, $\phi \rightarrow (1 + i\alpha^a t_r^a)\phi$

$\exists$ conj. rep. $\bar{r}$: $\phi^* \rightarrow (1 - i\alpha^a (t_r^a)^*)\phi^*$

$$t_{\bar{r}}^a = -(t_r^a)^* = -(t_r^a)^T$$

If $\exists$ matrix U, s.t. $t_{\bar{r}}^a = U t_r^a U^+$, then r is real.

Then, if η, ξ in same rep. r, then $G_{ab}\eta_a \xi_b$ is invariant.

For $G_{ab} = G_{ba}$, then r is strictly real,

for $G_{ab} = -G_{ba}$, then r is pseudoreal.

$SU(2)$: doublets: $V_a W_a$ is real.

$\epsilon^{ab} \eta_a \xi_b$ is pseudoreal.

In particle physics, we mainly work with fundamental reps, In particle physics, we mainly work with fundamental reps, anti-fundamental reps, and adjoint reps.

Fundamental of $SU(N)$ is N or $\square$. ($N\times 1$ - row vector)

Anti-fundamental of $SU(N)$ is $\bar{N}$ or $\bar{\square}$ ($1\times N$ - column vector).

Explicit reps (i.e. mappings)

Adjoint: Note f^{abc} provide representation matrices for adjoint rep.

$$(t^b_G)_{ac} = i f^{abc}$$

Explicit rep.: def. of $(t^a)_{ij}$...

This $(t^b_G)_{ac} = if^{abc}$ defines the generators for adjoint rep. G of $SU(N)$.

Adjoint is always real $\quad t^a_G = -(t^a_G)^*$

dimension of adjoint $= d(G) = N^2-1 \quad SU(N)$

$N(N-1)/2 \quad SO(N)$

$N(N+1)/2 \quad S_p(N)$

Concrete example:

Construct group invariant of $SU(2)$ contraction.

$$\mathcal{L} = \bar{L}_L i\not{D} L_L \ni (\bar{L}_L)^i \tau^a_{ij} W^a_\mu (L_L)_j + (\bar{L}_L)^i (L_L)_i$$

$L_L = P_L L$

$\bar{2} \times 2$ gives $SU(2)$ singlet.

$$\not{D} = \not{\partial} - ig\,\tau^a W^a_\mu \gamma^\mu$$

$$\tau^a = \frac{\sigma^a}{2} = \text{Pauli matrices.}$$

SU(2) indices written out,

$$L_L = \begin{pmatrix} \nu_L \\ \ell_L \end{pmatrix} \qquad \bar{L}_L = (\bar{\nu}_L \ \ \bar{\ell}_L)$$

$\nu_L = P_L \nu$

written out, not fermion indices.

χ_L
$\nu_L = P_L \nu$
$l_L = P_L l$

$$\mathcal{L} \supset (\bar{L}_L)_i g (\tau^a_{ij}) W^a_\mu \gamma^\mu (L_L)_j \star$$ i, j are $SU(2)$ indices + explicit.

Contrast w/ generic.

$$(\bar{L}_L)_i \, t^a_{ij} W^a \gamma^\mu (L_L)_j \star$$

But, in general, gauge fields transform in adj. rep.

So t^a_{ij} can use $(t^b)_{ac} = i f^{abc}$

$$[\sigma^a, \sigma^b] = 2 i \epsilon^{abc} \sigma^c$$

Convenient to use $\tau^a \equiv \frac{\sigma^a}{2}$ $\Rightarrow$ $\left[\frac{\sigma^a}{2}, \frac{\sigma^b}{2}\right] = i \epsilon^{abc} \frac{\sigma^c}{2}$

$$[\tau^a, \tau^b] = i \epsilon^{abc} \tau^c$$

In $SU(2)$, the f^{abc} are simply ϵ^{abc}, which I can use to explicitly write t^a_{ij}.

$$t^a_{ij} = i \epsilon^{iaj}$$

Gauge fields require adjoint rep. since they are from the connection of $\psi(y)$ and $\psi(x)$.

For any simple Lie algebra, $T^2 = t^a t^a$

(analogous to total spin J^2 in $SU(2)$)

commutes with all t^b.

$$t^a_r t^a_r \equiv C_2(r) \cdot \underbrace{\mathbb{1}}_{d(r) \times d(r) \text{ unit matrix.}}$$

$C_2(r)$ = quadratic Casimir

$$d(r) \, C_2(r) = d(G) \, C(r).$$

C_2 : quadratic Casimir

$$d(r)\, C_2(r) = d(G)\, C(r).$$

$$\Rightarrow \quad C(N) = \frac{1}{2}, \quad C_2(N) = \frac{N^2-1}{2N}$$

$$C(G) = N, \quad C_2(G) = N.$$
