Lecture 10, May 27, 2020

Wednesday, May 27, 2020 2:08 PM

Reminder: No lecture on June 1.
Lecture 11 on June 3.
HW 3 due on June 3.

Last time: Functional methods for scalar fields + fermionic fields.

Today: Begin unit on non-Abelian gauge theory.

Subtopics: Group theory + Lie Algebras
Quantization + Fadeev-Popov ghosts
Quantum Chromodynamics

Ref. Georgi - Lie Algebras in Particle Physics
Wu-Ki Tung - Group Theory in Physics.

From last time, we saw that fcnal. methods readily reproduce the known results of 2pt. + n-pt. correlation fcns. in free field theories. Then, a perturbative expansion of the action wrt. a perturbative coupling leads to the appropriate Feynman rule, where the fcnal. derivatives on the generating fcnal. extracted the appropriate external legs from vacuum.

The last issue is to use the fcnal. method for vector fields, but we will immediately run into a complication from gauge redundancy. So we will revisit this complication after studying non-Abelian gauge theory; will use fcnal method for quantization of non-Abelian gauge theories.

Recall QED:

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu}F^{\mu\nu} + \bar{\Psi}(i\not{\partial} - m)\Psi + \bar{\Psi} e \gamma^\mu A_\mu \Psi$$

Permits a local symmetry transformation, Permits a <u>local</u> symmetry transformation,

$$\psi(x) \to e^{i\alpha(x)}\psi(x).$$

If $\alpha(x) \equiv \alpha$, global phase.

Define derivative $\quad n^\mu \partial_\mu \psi(x) = \lim_{\epsilon \to 0} \frac{1}{\epsilon}\left[\psi(x+\epsilon n) - \psi(x)\right]$

Requires compensator since two spacetime points are involved.

$$\boxed{U(y,x) \to e^{i\alpha(y)}\, U(y,x)\, e^{-i\alpha(x)}}$$

scalar fcn. returns relative phase b/t y, x

Use $\boxed{U(x,x) \equiv 1}$, then $U(y,x) = \boxed{\exp\left[i\phi(y,x)\right]}$

Thus, $\psi(y)$ + $U(y,x)\psi(x)$ transform the same.

$$n^\mu D_\mu \psi(x) = \lim_{\epsilon \to 0} \frac{1}{\epsilon}\left[\psi(x+\epsilon n) - U(x+\epsilon n, x)\psi(x)\right]$$

defines covariant derivative.

We now expand $U(x+\epsilon n, x) \simeq 1 - ie\epsilon n^\mu A_\mu(x) + \mathcal{O}(\epsilon^2)$

$A_\mu(x)$ is connection

Then $\quad D_\mu \psi(x) = \partial_\mu \psi(x) + ieA_\mu \psi(x)$

and $\quad A_\mu(x) \to A_\mu(x) - \frac{1}{e}\partial_\mu \alpha(x)$

Derive $A_\mu(x) \to A_\mu(x) - \frac{1}{e}\partial_\mu \alpha(x)$

$$U(y,x) \to e^{i\alpha(y)}(1 - ie\epsilon n^\mu A_\mu)\, e^{-i\alpha(x)}$$

RHS

$$= e^{i\alpha(x+\epsilon n)}(1 - ie\epsilon n^\mu A_\mu)\, e^{-i\alpha(x)}$$

LHS

$$1 - ie\epsilon n^\mu A_\mu \to (1 + i\alpha(x+\epsilon n))(1 - ie\epsilon n^\mu A_\mu)(1 - i\alpha(x))$$

$$= (1 + i\alpha(x+\epsilon n) - i\alpha(x) - ie\epsilon n^\mu A_\mu + \mathcal{O}(\epsilon^2))$$

Match LHS + RHS at $\mathcal{O}(\epsilon)$:

$$-ien^\mu A_\mu = \frac{1}{\epsilon}\, i\left(\alpha(x+\epsilon n) - \alpha(x)\right) - ie\, n^\mu A_\mu(x)$$

$$-ie\, n^\mu A_\mu = i\frac{\partial \alpha(x)}{\partial x_\mu} - ie\, n^\mu A_\mu$$

$$A_\mu \to A_\mu - \frac{1}{e}\frac{\partial \alpha(x)}{\partial x_\mu} \qquad \checkmark$$

∂x_μ

$$A_\mu \to A_\mu - \frac{1}{e}\frac{\partial \alpha(x)}{\partial x_\mu} \quad ✓$$

Claim: $D_\mu \psi$ transforms same as ψ.

Pf: $D_\mu\psi = \partial_\mu\psi + ieA_\mu\psi$

$$\to \left[\partial_\mu + ie\left(A_\mu - \tfrac{1}{e}\partial_\mu\alpha\right)\right] e^{i\alpha}\psi$$

$$= e^{i\alpha}\partial_\mu\psi + ieA_\mu e^{i\alpha}\psi + e^{i\alpha}(i\partial_\mu\alpha)\psi - i(\partial_\mu\alpha)e^{i\alpha}\psi$$

$$= e^{i\alpha(x)}(D_\mu\psi)$$

$\bar\psi\psi$ is gauge inv. (+ Lorentz invariant)

+ $\bar\psi D_\mu\psi$ is gauge inv.

Then $D_\mu D_\nu\psi$ transforms as ψ.

And $[D_\mu, D_\nu]\psi$ also transforms as ψ.

But can also calculate

$$[D_\mu, D_\nu]\psi = \cancel{[\partial_\mu,\partial_\nu]}^{\,0}\psi + ie\left([\partial_\mu, A_\nu] - [\partial_\nu, A_\mu]\right)\psi - e^2\cancel{[A_\mu, A_\nu]}^{\,0}\psi$$

$-e^2(A_\mu A_\nu - A_\nu A_\mu)\psi$

$= -e^2(A_\mu A_\nu - A_\mu A_\nu)\psi$

$= 0$

$$= ie(\partial_\mu A_\nu - \partial_\nu A_\mu)\psi$$

$$\equiv ie F_{\mu\nu}\psi$$

Since ψ on RHS saturates gauge transformation $e^{i\alpha(x)}\psi$, then $\boxed{F_{\mu\nu}}$ is gauge invariant (in Abelian gauge theory).

$$ie\left((\partial_\mu A_\nu)\psi \underbrace{- A_\nu(\partial_\mu\psi)} - (\partial_\nu A_\mu)\psi + \underbrace{A_\mu(\partial_\nu\psi)}\right)$$

$\overset{?}{=} ie F_{\mu\nu}\psi$

$-A_\nu(\partial_\mu\psi) + A_\mu(\partial_\nu\psi)$

$-(\partial_\mu\psi)A_\nu + A_\mu(\partial_\nu\psi)$] IBP?

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Generalize to the situation where the local symmetry is not simply a phase.

G... (ie global symmetry) transformation simply a phase.

First, consider non-gauge (i.e. global symmetry) transformations

In other words, a <u>flavor symmetries</u>.

$$\mathcal{L} = \frac{-1}{4} F_{\mu\nu} F^{\mu\nu} + \sum_{i=1}^{N_F} \left(\overline{\Psi}_i \, (i\not{\partial} - m_i) \Psi_i + \overline{\Psi}_i Q_i e \gamma^\mu A_\mu \Psi_i \right)$$

Each Ψ_i has some charge Q_i

$\Psi \to e^{i Q_i \alpha(x)} \Psi$ under $U(1)$ trans.

If all $m_i = m$, $Q_i = Q$, then $U(N_F)$ flavor symmetry,

$U_{ij} \Psi_j = \Psi_i$, for $U \in U(N_F)$ matrix.

$$\sum_{i=1}^{N_F} (\overline{\Psi}_i \, i\not{\partial} \, \Psi_i) \xrightarrow{U(N_F)} \overline{\Psi}_j \, U^\dagger_{ji} \, i\not{\partial} \, U_{ij} \Psi_j = \overline{\Psi}_j \, i\not{\partial} \, \Psi_j$$

So this $\mathcal{L}$-term is invariant under $U(N_F)$ symmetry.

Make same argument for $m \overline{\Psi}_i \Psi_i + Q e \overline{\Psi}_i \gamma^\mu A_\mu \Psi_i$.

Natural generalization: Ψ_i transforms into Ψ_j as a spacetime dependent fcn.

$$U_{ij} \equiv \alpha^a(x) \, t^a_{ij}$$

$\hookrightarrow$ representation matrices.

Break until 3:08 pm.

The transformation $\Psi(x) \to V(x) \Psi(x)$ requires $V(x)$ to be unitary and linear (or antiunitary + antilinear, such as T reversal).

Assume $\mathcal{L}$ is invariant under set of infinitesimal transformations

$$\delta \Psi_l(x) = i \alpha^a(x) \, t^a_{lm} \Psi_m(x)$$

In order for this transformation to be close to identity, we must have any continuous symmetry be represented by

$$V(x) = \mathbb{1} + i\alpha^a(x) t^a + \mathcal{O}(\alpha^2).$$

with $\alpha^a(x)$ real, infinitesimal, t^a are Hermitian & linear.

The comparator generalizes from

$$U(y,x) \to e^{+i\alpha(y)}\, U(y,x)\, e^{-i\alpha(x)}$$

to

$$U(y,x) \to V(y)\, U(y,x)\, V^{\dagger}(x)$$

Then, $U(x+\epsilon n, x) = 1 + ig\epsilon n^\mu A^a_\mu t^a + \mathcal{O}(\epsilon^2)$

Furthermore, $n^\mu D_\mu \psi$ defined by

$$n^\mu D_\mu \psi = \lim_{\epsilon \to 0} \frac{1}{\epsilon}\left[\psi(x+\epsilon n) - (1 + ig\epsilon n^\mu A^a_\mu t^a)\psi(x)\right]$$

$$= n^\mu \partial_\mu \psi - n^\mu ig A^a_\mu t^a \psi$$

$$\Rightarrow D_\mu \psi = (\partial_\mu - ig A^a_\mu t^a)\psi$$

Again, comparator puts restrictions on t^a

$$U(x+\epsilon n, x) \to V(x+\epsilon n)\, U(x+\epsilon n, x)\, V^{\dagger}(x)$$

LHS RHS

$$1 + ig\epsilon n^\mu A^a_\mu t^a \Rightarrow (1 + i\alpha^b(x+\epsilon n)t^b)$$
$$(1 + ig\epsilon n^\mu A^a_\mu t^a)(1 - i\alpha^c(x)t^c)$$

Match at $\mathcal{O}(\epsilon)$

$$ig\epsilon n^\mu A^a_\mu t^a \Rightarrow i\alpha^b(x+\epsilon n)t^b - i\alpha^c(x)t^c$$
$$+ ig\epsilon n^\mu A^a_\mu t^a$$
$$- \alpha^b(x+\epsilon n)t^b g\epsilon n^\mu A^a_\mu t^a$$
$$+ g\epsilon n^\mu A^a_\mu t^a \alpha^c(x)t^c + \mathcal{O}(\alpha^2)$$
$$= ig\epsilon n^\mu A^a_\mu t^a + i(\alpha^a(x+\epsilon n) - \alpha^a(x))t^a$$
$$- g\epsilon n^\mu A^a_\mu (\alpha^b(x+\epsilon n)t^b t^a - \alpha^b(x)t^a t^b)$$

Take $\epsilon \to 0$.

$$ig A^a_\mu t^a \to ig A^a_\mu t^a + i\partial_\mu \alpha^a(x) t^a$$
$$\star\; - g A^a_\mu \alpha^b(x)(t^b t^a - t^a t^b)$$

Set + $[t^a, t^b]$ ...

$$\star\ -g A_\mu^a\, \alpha^b(x)\, (t^b t^a - t^a t^b)$$

Set ‡ $[t^a, t^b] = i f^{abc} t^c$

[or $[t^b, t^c] = i f^{bca} t^a = i f^{abc} t^a$
given f_{abc} totally antisymmetric under exchange of indices.]

‡ include Jacobi identity, will derive that f_{abc} is totally antisymmetric.

$$\begin{aligned}\star &= -g A_\mu^a\, \alpha^b(x)\, (t^b t^a - t^a t^b) \\ &= -g A_\mu^b\, \alpha^c(x)\, [t^c, t^b] \\ &= ig f^{abc} A_\mu^b \alpha^c t^a\end{aligned}$$

So $A_\mu^a \to A_\mu^a + \frac{1}{g}\partial_\mu \alpha^a + f^{abc} A_\mu^b \alpha^c$

Last term is non-vanishing in non-Abelian gauge theories, where we have a transformation of ψ that is matrix-valued.

$$\delta \psi_\ell(x) = i \alpha^a(x)\, t^a_{\ell m} \psi_m(x)$$

Again, calculate

$[D_\mu, D_\nu]\psi$

$$D_\nu \psi = \partial_\nu \psi - ig A_\nu^a t^a \psi$$

$$\begin{aligned}D_\mu D_\nu \psi = \partial_\mu \partial_\nu \psi &- ig (\partial_\mu A_\nu^a) t^a \psi - ig A_\nu^a t^a (\partial_\mu \psi) \\ &- ig A_\mu^a t^a (\partial_\nu \psi) - g^2 A_\mu^a t^a A_\nu^b t^b \psi\end{aligned}$$

Then, $[D_\mu, D_\nu]\psi$

$$\begin{aligned}&= -ig (\partial_\mu A_\nu^a) t^a \psi + ig (\partial_\nu A_\mu^a) t^a \psi \\ &\quad - g^2 A_\mu^a A_\nu^b (t^a t^b - t^b t^a)\psi\end{aligned}$$

Again, $[t^a, t^b] = i f^{abc} t^c$

$$[D_\mu, D_\nu]\psi = -ig(\partial_\mu A_\nu^a - \partial_\nu A_\mu^a) t^a \psi - g^2 A_\mu^a A_\nu^b\, i f^{abc} t^c \psi$$

So, $F_{\mu\nu}^a \equiv \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + g f^{abc} A^b A^c$

So, $F^a_{\mu\nu} \equiv \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^{abc} A^b_\mu A^c_\nu$

$+ \quad [D_\mu, D_\nu]\psi = -ig F^a_{\mu\nu} t^a \psi$

Remark: $F^a_{\mu\nu}$ is no longer gauge invariant by itself.

Exercise: $F^a_{\mu\nu} \to F^a_{\mu\nu} - f^{abc} \alpha^b F^c_{\mu\nu}$

Can read 15.3 in Peskin+Schroeder for discussion of Wilson loop + geometric interpretation of building gauge invariant theories + role of Stoke's thm.

Remark: $V(x) = \mathbb{1} + i\alpha^a(x) t^a + O(\alpha^2)$

The mathematics of possible objects that satisfy this requirement is known as Lie groups, which are special classes of continuous groups. t^a are generators of symmetry group.