08.128.165 Theorie 6a, Relativistische Quantenfeldtheorie
Quantum Field Theory I

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Homework set 8

Due December 21, 2022 by start of lecture.
Please note how long it took you to solve each problem.

8-1, 50 pts. *Practice with transcribing Feynman diagrams into matrix elements.* Use the following Feynman rules for interactions:

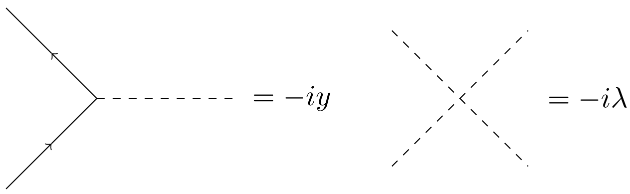


Figure 1: Feynman rules in Yukawa theory and $\lambda\phi^4/4!$ theory.

The Lagrangian is

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi)^2 - \frac{1}{2}m_s^2\phi^2 + \bar{\psi}(i\partial\!\!\!/)\psi - m_f\bar{\psi}\psi \tag{1}$$

$$- y\phi\bar{\psi}\psi - \frac{\lambda}{4!}\phi^4 \ . \tag{2}$$

With these momentum-space Feynman rules, write the corresponding matrix elements for the following diagrams. If the momenta are not labeled, be sure to label them appropriate in your diagram, ensuring momentum conservation at each vertex.

A, 5 pts. Scalar propagator correction in ϕ^4 theory.

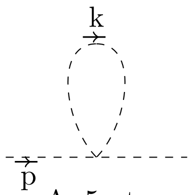
B, 5 pts. Scalar self-scattering in ϕ^4 theory.

C, 10 pts. Fermion propagator correction in Yukawa theory.

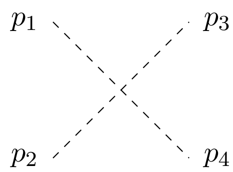
D, 10 pts. Scalar propagator correction in Yukawa theory.

E, 10 pts. Scalar scattering into fermions in Yukawa theory.

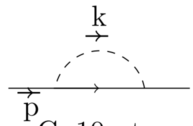
F, 10 pts. Fermion scattering into scalars in Yukawa theory.



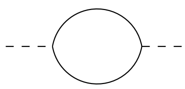
A, 5 pts



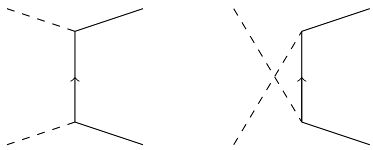
B, 5 pts



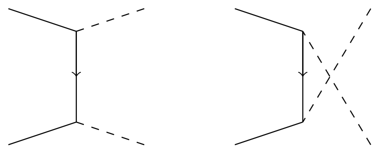
C, 10 pts.



D, 10 pts.



E, 10 pts.



F, 10 pts.

8-2, 30 pts. *Mandelstam variables.* For 2-to-2 scattering, with incoming momenta p_1 and p_2 and outgoing momenta p_3 and p_4, we can define the three Mandelstam kinematic invariants:

$$s \equiv (p_1 + p_2)^2 = (p_3 + p_4)^2 \tag{3}$$

$$t \equiv (p_1 - p_3)^2 = (p_4 - p_2)^2 \tag{4}$$

$$u \equiv (p_1 - p_4)^2 = (p_3 - p_2)^2 \ . \tag{5}$$

Here, p_1 and p_2 point into the collision vertex, while p_3 and p_4 point away from the collision vertex, and we require $p_1 + p_2 = p_3 + p_4$ for four-momentum conservation. *Aside: If you look at all one loop-level diagrams in $\lambda\phi^4$ theory, you can distinguish the s-channel, t-channel, and u-channel diagrams by the corresponding flow of the loop momenta.*

A, 15 pts. Given that the four external particles are on-shell, $p_i^2 = m_i^2$, derive the condition that $s + t + u = m_1^2 + m_2^2 + m_3^2 + m_4^2$.

B, 15 pts. In the center of mass frame, it is convenient to align p_1 and p_2 to be along the $\hat{z}$ axis and then set p_3 and p_4 back-to-back at an angle θ relative to the $+\hat{z}$ direction. An azimuthal rotation is sufficient to ensure p_3 and p_4 lie in the plane spanned by $\hat{x}$ and $\hat{z}$. Thus, with

$$p_1 = (E_1, |\vec{p}_1|\hat{z}) \ , p_2 = (E_2, -|\vec{p}_1|\hat{z}) \ , \tag{6}$$

$$p_3 = (E_3, |\vec{p}_3|\sin\theta, 0, |\vec{p}_3|\cos\theta) \ , p_4 = (E_4, -|\vec{p}_3|\sin\theta, 0, -|\vec{p}_3|\cos\theta) \ , \tag{7}$$

give explicit expressions for s, t, and u in terms of θ and the energies and masses of the four individual particles.

8-3, 20 pts. *Calculation shortcuts for matrix elements involving fermions.* When we consider cross sections involving fermions, we typically do not consider a specific spin orientation of either initial state or final state fermions. As a consequence, we will generally sum over possible spins of fermion spinors. We will thus make extensive use of the following identities.

$$\sum_s u^s(p)\bar{u}^s(p) = \not{p} + m\mathbf{1} \ , \tag{8}$$

$$\sum_s v^s(p)\bar{v}^s(p) = \not{p} - m\mathbf{1} \ . \tag{9}$$

Verify these identities by explicit calculation.

8-4, 20 pts. **Extra credit** *Kallen-Lehmann spectral density and matching one-particle states.* Read sections 10.2, 10.3 and 10.7 of Weinberg, Quantum Field Theory, Volume 1 (available on Moodle). In your own words, write a paragraph explaining the procedure to relate one-particle momentum states in the free theory to one-particle momentum states in an interacting theory. You can start with the Lehmann-Symanzik-Zimmermann reduction formula as given, but you should also explain the importance of concepts like poles in the amplitude, mass and wavefunction renormalization, and the possibility of non-one-particle states via the Källén-Lehmann spectral density. *Note: Weinberg uses the opposite sign metric compared to our convention in this course. We use* $g_{\mu\nu} = diag(1, -1, -1, -1)$*, while Weinberg uses* $g_{\mu\nu} = diag(-1, +1, +1, +1)$. *So, the four-vector dot products will all have the "wrong sign" compared to our notation.*