08.128.165 Theorie 6a, Relativistische Quantenfeldtheorie
Quantum Field Theory I

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Homework set 4

Due November 23, 2022 by the start of lecture.
Please note how long it took you to solve each problem.

4-1, 20 pts. Practice with Dirac algebra (part 2). Using the properties that $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$ and $\{\gamma^\mu, \gamma^5\} = 0$, as well as the fact that the trace of any number of γ matrices obeys cyclicity,

$$\mathrm{Tr}[\gamma^\mu \gamma^\nu \ldots \gamma^\sigma] = \mathrm{Tr}[\gamma^\nu \ldots \gamma^\sigma \gamma^\mu] \ , \tag{1}$$

evaluate the following Dirac trace identities (the trace is performed over the spin indices).

A, 4 pts. $\mathrm{Tr}[\gamma^\mu \gamma^\nu] = 4g^{\mu\nu}$,

B, 8 pts. $\mathrm{Tr}[\gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma] = 4(g^{\mu\nu} g^{\rho\sigma} - g^{\mu\rho} g^{\nu\sigma} + g^{\mu\sigma} g^{\nu\rho})$,

C, 4 pts. $\mathrm{Tr}[\gamma^{\mu_1} \gamma^{\mu_2} \ldots \gamma^{\mu_{2n-1}} \gamma^5] = 0$, (trace of an odd number of matrices with γ^5)

D, 4 pts. $\mathrm{Tr}[\gamma^{\mu_1} \gamma^{\mu_2} \ldots \gamma^{\mu_{2n-1}}] = 0$. (trace of an odd number of gamma matrices)

The two relations (C) and (D) are valid for any integer $n \geq 1$.

4-2, 10 pts. Practice with Dirac algebra (part 3). The charge conjugate matrix C has the properties $C\gamma^\mu C^{-1} = (-\gamma^\mu)^T$ and $C\gamma^5 C^{-1} = \gamma_5^T$. Show that

A, 5 pts. $\mathrm{Tr}[\gamma^\mu \gamma^\nu \gamma_5] = 0$,

B, 5 pts. $\mathrm{Tr}[\gamma^{\mu_1} \gamma^{\mu_2} \ldots \gamma^{\mu_{2n}}] = \mathrm{Tr}[\gamma^{\mu_{2n}} \gamma^{\mu_{2n-1}} \ldots \gamma^{\mu_1}]$.

4-3, 35 pts. Any complex 4×4 matrix M can be decomposed into Dirac basis, which are the 16 matrices $\Gamma^A \in \{1_{4\times 4}, i\gamma_5, \gamma^\mu, \gamma^\mu \gamma^5, \sigma^{\mu\nu}\}$, with $\mu, \nu = 0, 1, 2, 3$ and $\mu < \nu$. Thus we can write the expansion as

$$M = a 1_{4\times 4} + b i\gamma_5 + c_\mu \gamma^\mu + d_\mu \gamma^\mu \gamma_5 + \frac{1}{2} e_{\mu\nu} \sigma^{\mu\nu} \ , \tag{2}$$

with complex numbers a, …, $e_{\mu\nu}$. (We use a $\frac{1}{2}$ factor on the last term since the sum is not restricted to $\mu < \nu$.)

A, 20 pts. Define an associate set of 16 basis matrices with lower indices by

$$\Gamma_B \in \{1_{4\times 4}, -i\gamma_5, \gamma_\alpha, -\gamma_\alpha\gamma_5, \sigma_{\alpha\beta}\} \ , \tag{3}$$

for $\alpha, \beta = 0, 1, 2, 3$ with $\alpha < \beta$. Knowing the traces of different combinations of matrices, show that $\text{Tr}[\Gamma^A \Gamma_B] = 4\delta^A_B$ for any choices of A and B. Extract expressions for the coefficients $a, \ldots, e_{\mu\nu}$ in terms of traces of M with the matrices of the associated basis Γ_B. You can use the identity $\text{Tr}[\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\gamma_5] = -4i\epsilon^{\mu\nu\rho\sigma}$.

B, 15 pts. Using the above result, decompose the product $M = \gamma^\sigma\gamma^\rho\gamma^\delta$ into the Dirac basis.

4-4, 35 pts. Fierz identities. Let us consider products of bilinears, $(\bar{u}_1 M u_2)(\bar{u}_3 N u_4)$, where M and N are arbitrary 4×4 matrices and u_i are arbitrary, 4-component Dirac spinors. We can rewrite the bilinear product

$$(\bar{u}_1 M u_2)\,(\bar{u}_3 N u_4) = \sum_{A,B} C^A_B\,(\bar{u}_1 \Gamma_A u_4)\,(\bar{u}_3 \Gamma^B u_2) \tag{4}$$

where Γ_A and Γ^B are the 16 matrices of the Dirac basis and the associated basis in 5-3A.

A, 15 pts. Using the completeness of the 16 basis matrices, show that the coefficients C^A_B are given by

$$C^A_B = \frac{1}{16}\,\text{Tr}[\Gamma^A M \Gamma_B N] \ . \tag{5}$$

Hint: You can use the identity:

$$1_{ij} 1_{kl} = \frac{1}{4}\,(\Gamma_A)_{il}\,(\Gamma^A)_{kj} = \frac{1}{4}\,(\Gamma^B)_{il}\,(\Gamma_B)_{kj} \, , \tag{6}$$

where i, j, k, l are 4-component spinor indices running from 1 to 4 and $1 \equiv 1_{4\times 4}$ is the 4×4 identity matrix, so $1_{ij} = \delta_{ij}$. Given the above identity, you should calculate the quantity $M_{ij} N_{kl}$ where the indices correspond to given entries in the matrices M and N. To get started, write $M_{ij} N_{kl} = M_{ix} 1_{xj} 1_{ky} N_{yl}$ and use the identity above on the $1_{xj} 1_{ky}$ terms. To complete the problem, you will have the decompose M and N as sums over the basis matrices. *Aside: the identity relation is the full "completeness" relation that is referenced above. In particular, the identity relation shows that the Γ^A and Γ_B matrices form a bilinear, orthogonal basis; the only complication compared to normal vector spaces is that we do not have an inner product that gives 1 or 0, but instead we have a matrix multiplication that gives 4×4 matrices that are $1_{4\times 4}$ or 0. This is the same intuition from general $SU(2)$ matrices and the Pauli matrices as a basis.*

B, 10 pts. Work out the Fierz identity for

$$(\bar{u}_1 \gamma^\mu (1 - \gamma_5) u_2)(\bar{u}_3 \gamma_\mu (1 - \gamma_5) u_4) \ . \tag{7}$$

C, 10 pts. Work out the Fierz identity for

$$(\bar{u}_1 \gamma^\mu (1 - \gamma_5) u_2)(\bar{u}_3 \gamma_\mu (1 + \gamma_5) u_4) \ . \tag{8}$$